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Research Paper

Application of Linear Side Conditions $T\beta$=0 on Non-Full Rank Linear Models

Meliyana Bohori[1*], Mustofa Usman[1*], Riza Sawitri[1], Edwin Russel[1], Jamal Ibrahim Daoud[2]

[1] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Lampung, Bandar Lampung, 35141, Indonesia*
[2] *Department of Science in Engineering, Faculty of Engineering, International Islamic University, Selangor, 53100, Malaysia*
[*] **Corresponding author**: 2217031046@students.unila.ac.id

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Abstract
Linear model is one of the foundation of statistical analysis, including analysis of variance (ANOVA). However, under non-full rank conditions, the design matrix exhibits linear dependence, preventing the unique estimation of model parameters. This study aims to address this issue and to cope the problem by applying side conditions in the form of the linear constraint $T\beta$=0 without altering the model structure. The methods employed include the formulation of a restricted linear model, the derivation of parameter estimators, and simulation studies. The results show that side conditions produce unique, stable, and nearly unbiased parameter estimators. Furthermore, the results indicate that the power of test increases as differences between parameters increase and decreases as the variance of the error term increases. Thus, this approach is effective in ensuring the uniqueness of parameter estimates and improving the quality of analysis in ANOVA models with non-full rank design matrices.

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1. INTRODUCTION

A general linear model, $y = X\beta + \varepsilon$, is a fundamental concept in statistical analysis. In the analysis of factorial experimental designs, the design matrix X is often overparameterized, resulting in a design matrix that does not possess full-rank properties, i.e., Rank$(X) < p$ [1, 2, 3, 4]. This condition implies that the matrix $X'X$ is singular, so that the ordinary least squares estimator $\hat{\beta} = (X'X)^{-1}X'y$ cannot be uniquely determined, and the interpretation of the factor effects is not uniquely identifiable [3, 5].

A common approach to overcome this problem is the use of *generalized inverses* and model reparameterization [6, 7, 8, 9]. However, both approaches may produce non-unique estimators or alter the original structure of the model, making interpretation more difficult. As an alternative for resolving parameter non-identifiability in non-full-rank linear models, Seber [5] proposed the application of additional constraints, known as identifiability constraints. Similarly, Rencher and Schaalje [3] referred to these constraints as linear side conditions, which are linear restrictions imposed on the model parameters and expressed as $T\beta = 0$. Side conditions are introduced to compensate for rank deficiencies in the design matrix, where T is a $(p - r) \times p$ matrix with rank $p - r$. These side conditions are considered valid if they satisfy $M(X') \cap M(T') = \{0\}$, which ensures that $T\beta$ represents a set of non-estimable functions of β and serves as an identifiability constraint without affecting the estimated response values. Consequently, the model becomes well-defined and unique parameter estimation is possible [3, 5, 10].

Although the theoretical foundations of side conditions and recent studies have introduced augmented matrix techniques for handling non-full-rank models [3, 5, 10, 11, 12, 13, 14, 15], the existing literature does not provide explicit procedures for deriving parameter estimators. Furthermore, a comprehensive investigation of the statistical properties of these estimators has not yet been conducted. Therefore, a systematic study is needed to derive unique parameter estimators using linear side conditions and to investigate their statistical properties.

2. METHODS

The research steps taken in this study included a literature review on the application of linear constraints and side conditions to non-full-rank linear models. The additive linear model of the factorial completely randomized design is used by [1, 2, 16, 17, 18, 21] as the basis for the simulations and has generally been demonstrated in Equation (1) as follows:

$$y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk} \qquad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, c \end{cases} \tag{1}$$

Where in the form of a matrix, it can be written in Equation (2).

$$y = X\beta + \varepsilon \tag{2}$$

and the design matrix X is

$$X = [\mathbf{1}_a \otimes \mathbf{1}_b \otimes I_r \quad I_a \otimes \mathbf{1}_b \otimes \mathbf{1}_r \quad \mathbf{1}_a \otimes I_b \otimes \mathbf{1}_r]$$

For two-factor additive models with parameters

$$\beta' = [\mu, \alpha_1, \alpha_2, \ldots, \alpha_a, \beta_1, \beta_2, \ldots, \beta_b]$$

Equation (3) is often defined with more parameters than can be estimated, resulting in a design matrix X that is not full-rank which is shown in Equation (3).

$$X'X = \begin{bmatrix} abc_{1\times 1} & bc & ac_{1\times b} \\ bc_{a\times 1} & \mathrm{Diag}(bc)_{a\times a} & c_{a\times b} \\ abc_{b\times 1} & cb_{b\times a} & \mathrm{Diag}(ac)_{b\times b} \end{bmatrix} \tag{3}$$

This causes $X'X$ to be singular, so that $\hat{\beta} = (X'X)^{-1}X'y$ cannot be estimated uniquely. Therefore, linear side conditions are applied to ensure the model is full-rank.

2.1 Linear Side Conditions

In the linear model $y = X\beta + \varepsilon$, linear side conditions are a set of linear constraints applied to the model parameters to solve the problem of non-identifiability that arises when the design matrix X is not full rank and is written in the form $T\beta = 0$. Side conditions are applied to cover the rank deficiency in the design matrix X, where T is a matrix of size $(p - r) \times p$ with rank $p - r$ [5].

2.2 Conditional Estimability

Definition 2.1. Let c be an $n \times 1$ vector such that $E(a'y) = c'\beta$ for all β that satisfy $T\beta = 0$. Then $c'\beta$ is called a conditionally estimable function, and $a'y$ is a conditional unbiased estimate of $c'\beta$ [20].

Theorem 2.1. (Zimmerman [20]) Under the linear constraint $T\beta = 0$, the linear function $c'\beta$ can be estimated conditionally if and only if

$$c \in M(X' : T')$$

Based on the conditional estimability theorem [20], a linear function with parameters $c'\beta$ can be conditionally estimated under the constraint $T\beta = 0$ if and only if the vector c lies in the joint column space of the design matrix and the constraint matrix, that is $c \in M(X' : T')$.

This theorem shows that linear constraints serve to complete the information deficit caused by a design matrix that is not full rank, without altering the inferential meaning of the model. A linear constraint $T\beta = 0$ is called a valid side condition if it satisfies two conditions.

1. These constraints do not limit the information contained in the data, which is mathematically expressed as

$$M(X') \cap M(T') = \{0\}.$$

2. These constraints must be sufficient to ensure that all parameter functions can be estimated conditionally, that is

$$M(X' : T') = \mathbb{R}^p.$$

Once the first condition is met, by setting $rank(T) = (p - r)$ then

$$rank \begin{bmatrix} X \\ T \end{bmatrix} = rank(X) + rank(T)$$

$$rank \begin{bmatrix} X \\ T \end{bmatrix} = r + (p - r)$$

$$rank \begin{bmatrix} X \\ T \end{bmatrix} = p$$

so that it is fulfilled $M(X' : T') = \mathbb{R}^p$.

2.3 Power Test

In general, the hypotheses tested are formulated as follows:

$$H_0 : G\beta = g, \qquad H_a : G\beta \neq g.$$

where G is a $q \times p$ matrix of rank q, β is a $p \times 1$ parameter vector, and g is a $q \times 1$ vector [4]. The test criteria are:

$$\textit{Reject } H_0 \textit{ if } F_{calc} > F_\alpha.$$

In the process of hypothesis testing, there are two types of errors [21], as shown in Table 1.

3. RESULTS AND DISCUSSION

3.1 Parameter suspects with side conditions

Theorem 3.1 Let $y = X\beta + \varepsilon$, where X is an $n \times p$ matrix with $rank(X) = r < p$, and T is a $(p - r) \times p$ matrix with $rank(T) = p - r$, such that $T\beta$ is a set of non-estimable functions, then there exists exactly one vector $\hat{\beta}$ that satisfies Equation (4).

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Table 1. Hypothesis Testing Decision Matrix

Conclusion / H_0	**True Hypothesis**	**False Hypothesis**
Fail to Reject H_0	Correct Decision $(1-\alpha)$	Type II Error (β)
Reject H_0	Type I Error (α)	Correct Decision $(1-\beta)$

$$\begin{cases} y = X\beta + \varepsilon, \qquad E(\varepsilon) = 0,\ Cov(\varepsilon) = \sigma^2 I \\ T\beta = 0 \end{cases} \tag{4}$$

The system can be written in block matrix which is shown in Equation (5).

$$\begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ T \end{bmatrix} \beta + \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} \tag{5}$$

The Lagrange multiplier λ was used to minimize the constraints of the least-squares estimator for β, yielding the result shown in Equation (6).

$$\begin{aligned} L(\beta,\lambda) &= f(\beta) + \lambda g(\beta) \\ &= (y - X\beta)'(y - X\beta) + 2\lambda'(T\beta - 0) \\ &= y'y - 2y'X\beta + \beta'X'X\beta + 2\lambda'T\beta \end{aligned} \tag{6}$$

By taking a partial derivative and equalizing it by zero, the system obtains the constraints normal system as shown in Equation (7).

$$\begin{cases} \dfrac{\partial L(\hat{\beta},\hat{\lambda})}{\partial \beta} = -2X'y + 2X'X\hat{\beta} + 2T'\hat{\lambda} = 0, \\ \dfrac{\partial L(\hat{\beta},\hat{\lambda})}{\partial \lambda} = 2T'\hat{\beta} = 0 \end{cases} \tag{7}$$

After restructuring, the system of equations becomes a new system as shown in Equation (8).

$$\begin{cases} X'X\hat{\beta} + T'\hat{\lambda} = X'y \\ T\hat{\beta} = 0 \end{cases} \tag{8}$$

Equation (8) can be written in matrix form as presented in Equation (9).

$$\begin{bmatrix} X'X & T' \\ T & 0 \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} X'y \\ 0 \end{bmatrix} \tag{9}$$

Next, the solution for $\hat{\beta}$ and $\hat{\lambda}$ can be stated as in Equation (10).

$$\begin{bmatrix} \hat{\beta} \\ \lambda \end{bmatrix} = \begin{bmatrix} X'X & T' \\ T & 0 \end{bmatrix}^{-1} \begin{bmatrix} X'y \\ 0 \end{bmatrix} \tag{10}$$

To determine the block inverse of the matrix, the following theorem and corollary are used.

Theorem 3.2 (Wang & Chow [10])
Let S be a $p \times p$ symmetric matrix that is positive semidefinite and T be a $q \times p$ matrix. Then the general inverse of the matrix is given in Equation (11).

$$\begin{bmatrix} S & T' \\ T & 0 \end{bmatrix}^{-} = \begin{bmatrix} H^- - H^-T'Q^-TH^- & H^-T'Q^- \\ Q^-TH^- & Q^-Q - Q^- \end{bmatrix} \tag{11}$$

where $H = S + T'T$ and $Q = TH^-T'$.

Corollary 3.1
Let $S = X'X \geq 0$ be a $p \times p$ dimensional matrix, $M(X') \cap M(T') = \{0\}$ is satisfied, $rank(X) + rank(T) = p$, and $rank(T_{q\times p}) = q$ where $q = p - r$ then:
(i) $H = X'X + T'T$ and block matrices

$$\begin{bmatrix} X'X & T' \\ T & 0 \end{bmatrix}^{-1}$$

is a non-singular matrix.

(ii) $$\begin{bmatrix} X'X & T' \\ T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} H^{-1}X'XH^{-1} & H^{-1}T' \\ TH^{-1} & 0 \end{bmatrix}.$$

From the Corollary 3.1, we obtain a least-squares equation with constraints that has a unique solution given by Equation (12).

$$\begin{aligned} \begin{bmatrix} \hat{\beta} \\ \hat{\lambda} \end{bmatrix} &= \begin{bmatrix} (H)^{-1}X'X(H)^{-1} & (H)^{-1}T' \\ T(H)^{-1} & 0 \end{bmatrix} \begin{bmatrix} X'y \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} (H)^{-1}X'X(H)^{-1}X'y \\ T(H)^{-1}X'y \end{bmatrix} \\ &= \begin{bmatrix} (H)^{-1}X'y \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} (X'X + T'T)^{-1}X'y \\ 0 \end{bmatrix} \end{aligned} \tag{12}$$

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In a matrix form, that estimator can be written as presented in Equation (13).

$$\hat{\beta} = \begin{bmatrix} X' \\ T \end{bmatrix}' \left(\begin{bmatrix} X' \\ T \end{bmatrix} \begin{bmatrix} X' \\ T \end{bmatrix}' \right)^{-1} \begin{bmatrix} y \\ 0 \end{bmatrix} \tag{13}$$

For factorial design, the design matrix can be written as in Equation (14).

$$\begin{bmatrix} X' \\ T \end{bmatrix}' \begin{bmatrix} X' \\ T \end{bmatrix} = \begin{bmatrix} abc_{1\times 1} & bc_{1\times a} & ac_{1\times b} \\ bc_{a\times 1} & \mathrm{Diag}(bc)_{a\times a} + I_{a\times a} & c_{a\times b} \\ ac_{b\times 1} & cb_{b\times a} & \mathrm{Diag}(ac)_{b\times b} + I_{b\times b} \end{bmatrix} \tag{14}$$

Which is non-singular. Equation (12) and (13) are similar, because:

1. $X'X(H)^{-1}X' = X',\ T(H)^{-1}X' = 0.$

To prove (1) and (2), let X_1 be a $k \times p$ matrix containing linearly independent rows of X, and define

$$L = \begin{bmatrix} X_1 \\ T \end{bmatrix}.$$

Then L is a nonsingular $p \times p$ matrix. There exists an $n \times k$ matrix C such that $X = CX_1 = (C, 0)L$. In addition, it can be written as $T = (0, I_q)L$.

$$X'X = L' \begin{bmatrix} C'C & 0 \\ 0 & 0 \end{bmatrix} L, \quad \text{and} \quad T'T = L' \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix} L.$$

Proof (i)

$$\begin{aligned} X'X(X'X + T'T)^{-1}X' &= L' \begin{bmatrix} C'C & 0 \\ 0 & 0 \end{bmatrix} L \\ &\quad \times \left(L' \begin{bmatrix} C'C & 0 \\ 0 & I_q \end{bmatrix} L \right)^{-1} X' \\ &= L' \begin{bmatrix} C'C & 0 \\ 0 & 0 \end{bmatrix} L \\ &\quad \times L^{-1} \begin{bmatrix} (C'C)^{-1} & 0 \\ 0 & I_q \end{bmatrix} (L')^{-1} X' \\ &= L' \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} (L')^{-1} X' \\ &= L' \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} (L')^{-1} L' \begin{bmatrix} C' \\ 0 \end{bmatrix} \\ &= L' \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C' \\ 0 \end{bmatrix} \\ &= L' \begin{bmatrix} C' \\ 0 \end{bmatrix} = X'. \end{aligned}$$

Therefore $X'X(H'H)^{-1}X' = X'$. In particular, this is equivalent to choosing $(H'H)^{-1}$ as the generalized inverse of $X'X$.

Proof (ii)

$$\begin{aligned} T(X'X + T'T)^{-1}X' &= (0, I_q)L \\ &\quad \times \left(L^{-1} \begin{bmatrix} (C'C)^{-1} & 0 \\ 0 & I_q \end{bmatrix} (L')^{-1} \right) \\ &\quad \times L' \begin{bmatrix} C' \\ 0 \end{bmatrix} \\ &= 0. \end{aligned}$$

3.2 Checking the Minimum Variance and Bias of the Estimator β

The estimator generated by adding the side condition $T\beta = 0$ is an unbiased estimator of β if $E(\hat{\beta}) = \beta$. The unbiasedness test is performed by calculating its expectation as in Equation (15).

$$\begin{aligned} E(\hat{\beta}) &= \left(\begin{bmatrix} X' \\ T \end{bmatrix}' \begin{bmatrix} X' \\ T \end{bmatrix} \right)^{-1} \begin{bmatrix} X' \\ T \end{bmatrix} E\left(\begin{bmatrix} Y \\ 0 \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} X' \\ T \end{bmatrix}' \begin{bmatrix} X' \\ T \end{bmatrix} \right)^{-1} \begin{bmatrix} X' \\ T \end{bmatrix}' \begin{bmatrix} X \\ T \end{bmatrix} \beta = \beta \end{aligned} \tag{15}$$

Thus, the estimator is unbiased because it satisfies $E(\hat{\beta}) = \beta$.

Since $\hat{\beta}$ is a linear transformation of y in the constrained model in Equation (5), the covariance matrix is

$$var \begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma^2 I_n & 0 \\ 0 & 0 \end{bmatrix}.$$

The variance of the estimator is computed using the following linear transformation property which is shown in Equation (16).

$$\begin{aligned} var(\hat{\beta}) &= \left(\begin{bmatrix} X' \\ T \end{bmatrix}' \begin{bmatrix} X' \\ T \end{bmatrix} \right)^{-1} \begin{bmatrix} X' \\ T \end{bmatrix}' \begin{bmatrix} \sigma^2 I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X' \\ T \end{bmatrix} \left(\begin{bmatrix} X' \\ T \end{bmatrix}' \begin{bmatrix} X' \\ T \end{bmatrix} \right)^{-1} \\ &= \sigma^2 (X'X + T'T)^{-1} X'X (X'X + T'T)^{-1} \\ &= \sigma^2 H^{-1} X'X H^{-1}. \end{aligned} \tag{16}$$

To prove that $\hat{\beta}$ is the Best Linear Unbiased Estimator (BLUE) under the constraint $T\beta = 0$, an alternative estimator is introduced

$$\hat{\theta} = \left((\tilde{X}'\tilde{X})^{-1}\tilde{X}' + A \right) \tilde{y}$$

[18].

For convenience, the following additional notation is introduced

$$\tilde{X} = \begin{bmatrix} X \\ T \end{bmatrix}, \qquad \tilde{y} = \begin{bmatrix} y \\ 0 \end{bmatrix}, \qquad \Sigma = \mathrm{var}(\tilde{y}) = \begin{bmatrix} \sigma^2 I_n & 0 \\ 0 & 0 \end{bmatrix},$$

$$M = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'.$$

Therefore, $\hat{\theta} = (M + A)\tilde{y}$.

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Unbiasedness $\hat{\theta}$

To ensure that the estimator $\hat{\theta}$ remains unbiased, we examine its expected value $E(\hat{\theta}) = (M + A)\tilde{X}\beta = \beta + A\tilde{X}\beta$. For the expected return to equal β, the following condition must be true $A\tilde{X} = 0$ (or by component $AX = 0, \quad AT = 0$).

Variance $Var(\hat{\theta})$

Using the properties of linear transformations:

$$
\begin{aligned}
Var(\hat{\theta}) &= (M + A)\Sigma(M + A)' \\
&= M\Sigma M' + M\Sigma A' + A\Sigma M' + A\Sigma A' \\
&= var(\hat{\beta}) + \sigma^2 AA'.
\end{aligned}
$$

where AA' is the additional variance resulting from the modification of the estimation. The i-th element of the main diagonal of the matrix AA' is

$$\sum_{i=1}^{n} a_{ii}^2 \geq 0, \qquad i = 1, 2, \ldots, n.$$

The non-negativity of the squared elements in matrix A establishes $\hat{\beta}$ as the Best Linear Unbiased Estimator (BLUE) under the linear restriction $T\beta = 0$.

3.3 Estimating Parameter σ^2

In a model with a non-full-rank design matrix, the estimator of the error variance is given by Equation (17).

$$s^2 = \frac{SS_{Res}}{n - r} \tag{17}$$

Let n denote the total sample size and r denote the rank of X. We will show that s^2 is an unbiased estimator of σ^2. Note that

$$E(s^2) = \frac{1}{n-r}\left(\begin{bmatrix} y' & 0 \end{bmatrix}\left[I - \begin{bmatrix} X \\ T \end{bmatrix}\left(\begin{bmatrix} X' \\ T \end{bmatrix}'\begin{bmatrix} X' \\ T \end{bmatrix}\right)^{-1}\begin{bmatrix} X' \\ T \end{bmatrix}'\right]\begin{bmatrix} y \\ 0 \end{bmatrix}\right).$$

An alternative notation is used with $P = \tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}'$, then

$$
\begin{aligned}
E(s^2) &= \frac{1}{n-r}E\left(\tilde{y}'(I - P)\tilde{y}\right) \\
&= \frac{1}{n-r}\left(tr((I - P)\Sigma) + \mu'(I - P)\mu\right) \\
&= \frac{1}{n-r}\left(tr((I - P)\Sigma) + (\tilde{X}\beta)'(I - P)(\tilde{X}\beta)\right) \\
&= \frac{1}{n-r}tr\left((I - P)\begin{bmatrix} \sigma^2 I_n & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \frac{\sigma^2}{n-r}tr\left(I_n - X'(\tilde{X}'\tilde{X})^{-1}\right) \\
&= \frac{\sigma^2}{n-r}(n - r) \\
&= \sigma^2.
\end{aligned}
$$

Therefore, s^2 is an unbiased estimates of σ^2.

3.4 Simulation

The data in this study were generated via simulation using SAS 9.4 to develop a $\hat{\beta}$ estimator program with the application of the linear constraint $T\beta = 0$. The simulation design included two factors: A with 4 levels and B with 3 levels, each repeated 3 times, resulting in 36 observations.

3.4.1 Simulation Bias Testing

In additive form, the model used:

$$y = 10 - a_1 + 0.5a_2 + 0.3a_3 + 0.2a_4 - 0.5\beta_1 + 0.4\beta_2 + 0.1\beta_3 + \varepsilon$$

with $\varepsilon \sim N(0, 1)$. To assess unbiasedness, $\hat{\beta}$ is compared against the true parameters using 1000 Monte Carlo simulations to ensure statistical stability.

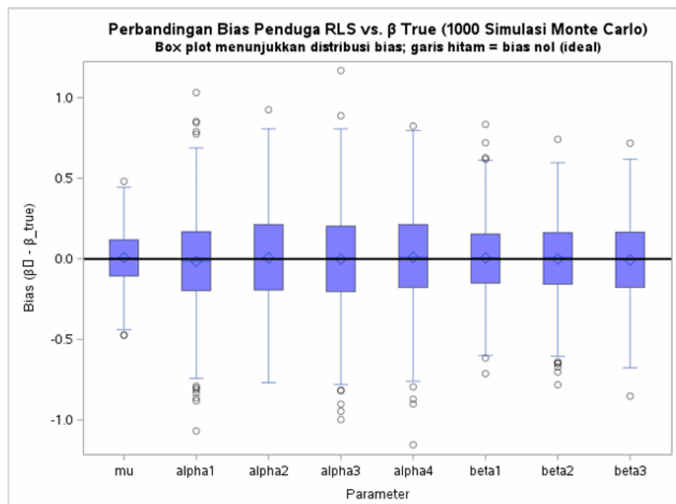

Figure 1. The Comparison of Bias for RLS Estimation and True β

Figure 1 shows that the average bias of each parameter is very close to zero, indicating that the RLS estimator with side conditions $T\beta$=0 is nearly unbiased overall. This is consistent with the boxplot in Figure 1, where the bias distribution of each parameter tends to be centered on the zero line (ideal).

3.4.2 Power of Test

The F_{calc} used in this simulation is presented in Equation (18).

$$F_{calc} = \frac{\dfrac{SS_{res}(\text{full}) - SS_{res}(\text{reduce})}{r - r_0}}{\dfrac{SS_{res}^{full}}{n - r}} \tag{18}$$

For the test, a thousand data sets were randomly generated, and their test power was calculated. This test was conducted using different values of σ $\{2, 4, 6, 9\}$ to examine how the test power varies across different levels of data variance.

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Table 2. The Power of Test for Difference Values of σ for Hypothesis 1

Iterations	Beta								σ			
	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	1	2	4	6
1	0	0	0	0	0	0	0	0	0.053	0.053	0.053	0.053
2	0	-0.600	0.200	0.180	0.220	0	0	0	0.220	0.086	0.068	0.059
3	0	-1.200	0.400	0.360	0.440	0	0	0	0.732	0.220	0.118	0.081
4	0	-1.800	0.600	0.540	0.660	0	0	0	0.980	0.454	0.220	0.118
5	0	-2.400	0.800	0.720	0.880	0	0	0	1.000	0.732	0.368	0.187
6	0	-3.000	1.000	0.900	1.100	0	0	0	1.000	0.905	0.565	0.271
7	0	-3.600	1.200	1.080	1.320	0	0	0	1.000	0.980	0.732	0.368
8	0	-4.200	1.400	1.260	1.540	0	0	0	1.000	0.996	0.856	0.487
9	0	-4.800	1.600	1.440	1.760	0	0	0	1.000	1.000	0.941	0.616
10	0	-5.400	1.800	1.620	1.980	0	0	0	1.000	1.000	0.980	0.732
11	0	-6.000	2.000	1.800	2.200	0	0	0	1.000	1.000	0.995	0.821
12	0	-6.600	2.200	1.980	2.420	0	0	0	1.000	1.000	0.999	0.894
13	0	-7.200	2.400	2.160	2.640	0	0	0	1.000	1.000	1.000	0.941
14	0	-7.800	2.600	2.340	2.860	0	0	0	1.000	1.000	1.000	0.970
15	0	-8.400	2.800	2.520	3.080	0	0	0	1.000	1.000	1.000	0.989
16	0	-9.000	3.000	2.700	3.300	0	0	0	1.000	1.000	1.000	0.995
17	0	-9.600	3.200	2.880	3.520	0	0	0	1.000	1.000	1.000	0.997
18	0	-10.200	3.400	3.060	3.740	0	0	0	1.000	1.000	1.000	1.000
19	0	-10.800	3.600	3.240	3.960	0	0	0	1.000	1.000	1.000	1.000
20	0	-11.400	3.800	3.420	4.180	0	0	0	1.000	1.000	1.000	1.000

Table 3. The Power of Test for Difference Values of σ for Hypothesis 2

Iteration	Beta								σ			
	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	1	2	4	6
1	0	0	0	0	0	0.000	0.000	0.000	0.050	0.050	0.050	0.050
2	0	0	0	0	0	-0.400	0.126	0.274	0.205	0.089	0.069	0.059
3	0	0	0	0	0	-0.626	0.260	0.366	0.423	0.143	0.091	0.071
4	0	0	0	0	0	-0.774	0.311	0.463	0.584	0.194	0.112	0.084
5	0	0	0	0	0	-1.126	0.403	0.723	0.910	0.366	0.184	0.108
6	0	0	0	0	0	-1.503	0.580	0.923	0.992	0.560	0.298	0.151
7	0	0	0	0	0	-1.764	0.693	1.071	0.999	0.709	0.386	0.196
8	0	0	0	0	0	-1.953	0.764	1.189	1.000	0.794	0.461	0.223
9	0	0	0	0	0	-2.286	0.983	1.303	1.000	0.907	0.567	0.305
10	0	0	0	0	0	-2.587	1.109	1.478	1.000	0.974	0.679	0.370
11	0	0	0	0	0	-2.840	1.252	1.588	1.000	0.990	0.763	0.429
12	0	0	0	0	0	-3.142	1.483	1.659	1.000	0.994	0.843	0.498
13	0	0	0	0	0	-3.452	1.667	1.785	1.000	0.997	0.903	0.575
14	0	0	0	0	0	-3.666	1.764	1.902	1.000	0.999	0.950	0.629
15	0	0	0	0	0	-3.920	1.865	2.055	1.000	1.000	0.975	0.686
16	0	0	0	0	0	-4.192	1.966	2.226	1.000	1.000	0.985	0.749
17	0	0	0	0	0	-4.473	2.058	2.415	1.000	1.000	0.993	0.802
18	0	0	0	0	0	-4.815	2.365	2.450	1.000	1.000	0.994	0.859
19	0	0	0	0	0	-5.103	2.507	2.596	1.000	1.000	0.997	0.892
20	0	0	0	0	0	-5.396	2.520	2.876	1.000	1.000	0.999	0.935

a. Results of the Simulation for Hypothesis 1
   Hypothesis 1 was tested using $\alpha = 0.05$, with the hypothesis under test being

$$H_0 \,:\, a_i = 0$$

$$H_a \,:\, \text{There is at least one } a_i \neq 0$$

The movement of the test power value on hypothesis 1 can be seen in the form of a graph as follows.

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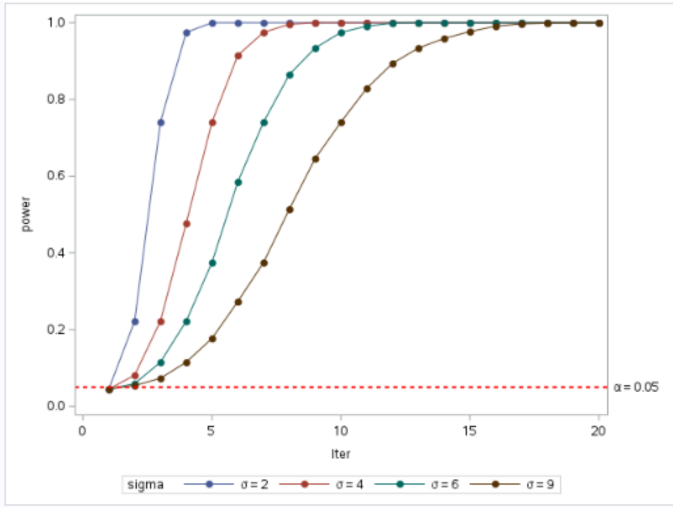

Figure 2. Power of the Test for Difference Values of σ for Hypothesis 1

b. Results of the Simulation for Hypothesis 2

Hypothesis 1 was tested using $\alpha = 0.05$, with the hypothesis under test being

$$H_0 : a_i = 0$$

$$H_a : \text{There is at least one } a_i \neq 0$$

The movement of the test power value on hypothesis 2 can be seen in the form of a graph as follows.

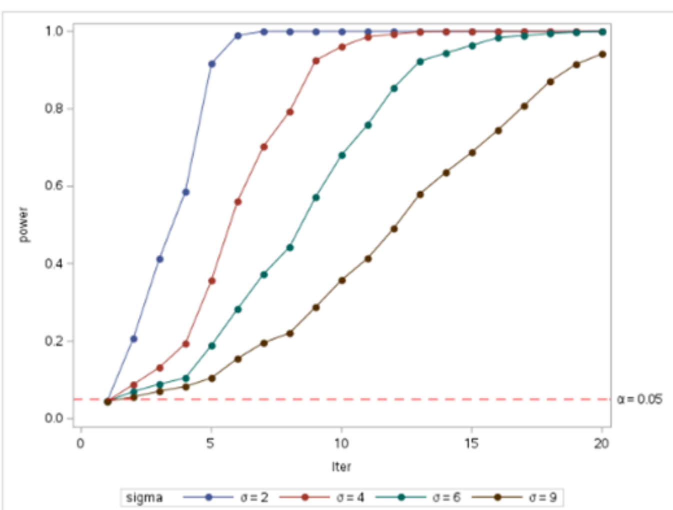

Figure 3. Power of the Test for Difference Values of σ for Hypothesis 2

Based on the tables and graphs from the two hypothesis tests above, it can be seen that in the first iteration, the test power is the lowest for all variance values $\sigma = 2$, $\sigma = 4$, $\sigma = 6$ and $\sigma = 9$. This occurs because the parameter vector used in the first iteration

$$\beta' = (0, 0, 0, 0, 0, 0, 0, 0, 0)',$$

which demonstrates that there is no significant difference among the levels of factor A, so this result is very close to the null hypothesis.

As a result, the probability of rejecting H_0 becomes very small, as reflected by the low test power. As the number of iterations increases, the values of the components in the β vector move further away from zero, meaning that the differences between the factor levels become larger.

This causes the distance between the parameters and the null hypothesis to increase, thereby also increasing the probability of rejecting H_0 when H_0 is false. This can be seen clearly from the increase in the test statistic's p-value, which approaches 1 as the number of iterations increases for all values of σ. In addition, the table also shows the effect of the error variance. For each iteration, the following pattern holds Power ($\sigma = 2$) > Power ($\sigma = 4$) > Power ($\sigma = 6$) > Power ($\sigma = 9$). This means that as the error variance increases, the test's ability to detect differences between treatments decreases, resulting in lower test power. Thus, these results confirm that the test power for Hypothesis 1 increases as the effect size (the distance β from H_0) increases and decreases as the error variance σ^2 increases. This pattern indicates that the RLS-based testing procedure with the constraint $\mathbf{T}\boldsymbol{\beta} = 0$ behaves consistently with testing theory in restricted linear models.

4. CONCLUSIONS

Based on the results and discussion, it can be concluded that: The application of side conditions can solve the non-full-rank problem by restricting the parameter space so that the model parameters can be obtained uniquely without altering the column space structure of the design matrix. Simulation results show that the application of the side condition $T\beta = 0$ successfully produces stable and nearly unbiased parameter estimators and provides good testing performance, where the test power increases as the parameter difference grows larger and decreases as the error variance increases.

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