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Research Paper

The Use of Model Reduction Method in the Combined Analysis of Two Randomized Complete Block Designs

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Abstract
Randomized Complete Block Design (RCBD) is an experimental design frequently used to control variability due to confounding factors. This design groups experimental units into relatively homogeneous blocks. However, the RCBD mathematical model contains linear constraints on the treatment and block effect parameters, which causes the design matrix to be non-full rank. Structural problems become more complex when combining two RCBD models in one analytical framework to expand information. This study applies the Model Reduction Method (MRM) to the combination of two RCBD models, focusing on the formation of a full rank model, parameter estimation using the Least Squares method, and evaluation of the estimator properties based on the Best Linear Unbiased Estimator (BLUE) criteria. This study also includes simulations using SAS 9.4 software to assess the performance of parameter estimation and hypothesis testing. The results demonstrate that MRM is effective in transforming a non-full rank model into a full rank model. Parameter estimation in the full rank model produces an estimator that is unbiased and has minimum variance, in accordance with the BLUE criteria. The results of the hypothesis testing show that the model obtained has good test power under low error variance conditions, and continues to show consistent performance reaching a value of one under all error variance conditions as the parameter distribution increases.

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1. INTRODUCTION

The general linear model is a basic framework that is widely used to data analysis, especially on data from various experimental and observational settings that aim to study the effect of treatment on a response variable [1, 2, 3, 4]. The general linear model is expressed in the form $Y = X\beta + \varepsilon$, with interpretation and statistical properties that depend on the assumptions of the error distribution, the structure of the covariance matrix and the rank of the design matrix structure [5, 6, 7]. Various experimental designs can be viewed as special cases of the general linear model that have a specific design matrix structure and parameter constraints.

Randomized Complete Block Design (RCBD) is an experimental design frequently used to control variability due to block factors. The RCBD design groups experimental units into relatively homogeneous blocks so that unwanted variations can be minimized [8, 9, 10, 11, 12, 13]. The use of RCBD increases the accuracy of analysis of treatment effects and is widely applied in research in agriculture, biology, and applied sciences [14, 15, 16]. The RCBD model is stated in Equation (1) as follows:

$$y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij}, \qquad \begin{cases} i = 1, 2, \ldots, b, \\ j = 1, 2, \ldots, t. \end{cases} \tag{1}$$

where y_{ij} is the response in the i-th block and the j-th treatment. μ is the general mean. β_i is the effect of the i-th block. τ_j

is the effect of the j-th treatment. ε_{ij} is the error in the i-th block and the j-th treatment $\varepsilon_{ij} \sim i.i.d.\, N(0, \sigma^2)$.

RCBD has linear constraints on the treatment effect parameters $\left(\sum_{i=1}^{t} \tau_i = 0\right)$ and block effects $\left(\sum_{i=1}^{b} \beta_i = 0\right)$, which causes the design matrix to be non-full rank or the number of parameters exceeds the rank of the matrix [5, 17, 18, 19]. Structural problems become more complex when two RCBD models are combined to improve analysis efficiency. The combined structure produces parameters that are linearly interdependent, so that the parameter estimates are not unique [20, 21, 22]. Hocking [3] introduced the Model Reduction Method (MRM) as an approach to reduce a non-full rank model containing constraints to a full rank model without constraints through matrix transformation.

2. METHODS

Suppose given a linear model as presented in Equation (2)

$$Y = X\theta + \varepsilon \tag{2}$$

with Y is a response vector of size $n \times 1$. X is design matrix of size $n \times p$ with rank $< p$. θ is parameter vector of size $p \times 1$. ε is an error vector of size $n \times 1$ and $\varepsilon \sim N(0, \sigma^2 I)$. The parameter constraints is given in Equation (3).

$$G\theta = g \tag{3}$$

where G is a matrix of size $q \times p$ of rank q, g is a constant vector of size $q \times 1$.

The initial step in MRM is to define an orthogonal permutation matrix T which aims to rearrange the parameters so that the constraints can be separated. Applying the transformation to Equation (2) produces Equation (4) as follows:

$$Y = X_1\theta_1 + \varepsilon \tag{4}$$

with $X_1 = XT'$ and $\theta_1 = T\theta$. The parameter constraints after transformation is given in Equation (5).

$$G_1\theta_1 = g \tag{5}$$

Assume that θ_1 and G_1 can be partitioned, so that:

$$G_{11}\theta_{11} + G_{12}\theta_{12} = g$$

G_{11} is a matrix $q \times q$ with rank q, so that:

$$\theta_{11} = G_{11}^{-1} g - G_{11}^{-1} G_{12}\theta_{12}$$

The design matrix X_1 is partitioned into $X_1 = [X_{11} \;\; X_{12}]$ and substituted into Model (4), so that we obtain Equation (6).

$$Y_r = X_r\theta_r + \varepsilon \tag{6}$$

with:

$$\begin{aligned} Y_r &= Y - X_{11}G_{11}^{-1} g, \\ X_r &= X_{12} - X_{11}G_{11}^{-1} G_{12}, \\ \theta_r &= \theta_{12}. \end{aligned}$$

2.1 Model Transformation Using MRM

The first transformation in Model (8): Equation (6) is called the full rank model without constraints [3].

3. RESULT AND DISCUSSIONS

RCBD model for two designs is given in Equation (7).

$$y_{ij(l)} = \mu_{(l)} + \beta_{i(l)} + \tau_{j(l)} + \varepsilon_{ij(l)}, \qquad \begin{cases} i = 1, 2, \ldots, b, \\ j = 1, 2, \ldots, t, \\ l = 1, 2. \end{cases} \tag{7}$$

y_{ijl} is the response in the i-th block, the j-th treatment and the l-th design. μ_l is the general mean of the l-th design. β_{il} is the effect of the i-th block in the l-th design. τ_{jl} is the effect of the j-th treatment in the l-th design. ε_{ijl} is the error in the i-th block, the j-th treatment and the l-th design. $\varepsilon_{ijl} \sim i.i.d.\, N(0, \sigma^2)$. The constraints on the parameters are: $\sum_{i=1}^{b} \beta_{il} = 0$, $\sum_{i=1}^{t} \tau_{jl} = 0$, $l = 1, 2$. The combined model of two RCBDs in the form of a matrix model is given in Equation (8).

$$Y = \Gamma\theta + \Psi \tag{8}$$

Combined design matrix:

$$\Gamma = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} = diag(X_1, X_2)$$

with $X_l = [\mathbf{1}_n \; E \; F]$, $E = I_b \otimes \mathbf{1}_t$, and $F = \mathbf{1}_b \otimes I_t$.
Combined parameter matrix:

$$\theta = \begin{bmatrix} (\mu_1 \; \beta_1' \; \tau_1')' \\ (\mu_2 \; \beta_2' \; \tau_2')' \end{bmatrix}$$

with $\beta_l' = (\beta_{1l} \; \beta_{2l} \; \cdots \; \beta_{bl})$ and $\tau_l' = (\tau_{1l} \; \tau_{2l} \; \cdots \; \tau_{tl})$.
Combined error matrix:

$$\Psi = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \sim N(0, \sigma^2 I_{2n})$$

$$\varepsilon_l' = (\varepsilon_{11l} \; \varepsilon_{121} \; \varepsilon_{131} \; \cdots \; \varepsilon_{btl})$$

Assuming the $\mathrm{Var}(\varepsilon_l) = \sigma_l^2 I_n$, so that the error vector variance combination is

$$\mathrm{Var}(\Psi) = diag(\sigma_1^2 I_n, \sigma_2^2 I_n)$$

The parameter constraints is given in Equation (9).

$$\Omega\theta = 0 \tag{9}$$

with

$$\Omega = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} = diag(G_1, G_2)$$

$$G_l = \begin{bmatrix} 0 & \mathbf{1}_b' & \mathbf{0}_t' \\ 0 & \mathbf{0}_b' & \mathbf{1}_t' \end{bmatrix}.$$

The combined RCBD model stated in Equation (8) is a non-full rank model. Based on the assumption connectedness, it is obtained that the rank of the combined design matrix Γ is $2(b+t-1)$, which is smaller than the number of parameters $2(b+t+1)$ [16].

3.1 Model Transformation Using MRM

The first transformation in Model (8):

$$Y = \Gamma T' T\theta + \Psi$$

equivalent to Equation (10).

$$Y = \Gamma^* \Theta^* + \Psi \tag{10}$$

where

$$\begin{aligned} \Gamma^* &= \Gamma T' = diag[(E \;\; F \;\; \mathbf{1}_n), (E \;\; F \;\; \mathbf{1}_n)], \\ \Theta^* &= T\theta = (\beta_l' \; \tau_l' \; \mu_l)', \\ \Psi &= [\varepsilon_1' \;\; \varepsilon_2']'. \end{aligned}$$

Equation (11) shows parameter constraints.

$$\Omega^* \Theta^* = 0 \tag{11}$$

where

$$\Omega^* = \begin{bmatrix} G_1^* & 0 \\ 0 & G_2^* \end{bmatrix} = diag(G_1^*, G_2^*).$$

$$G_l^* = \begin{bmatrix} \mathbf{1}_b' & \mathbf{0}_t' & 0 \\ \mathbf{0}_b' & \mathbf{1}_t' & 0 \end{bmatrix}.$$

The second transformation on Equation (10):

$$Y = \Gamma^* \Lambda' \Lambda \Theta^* + \Psi$$

equivalent to Equation (12).

$$Y = \Gamma_1^* \Theta_1^* + \Psi \tag{12}$$

with:

$$\begin{aligned} \Gamma_1^* &= \Gamma^* \Lambda' = diag\left[(E \;\; F \;\; \mathbf{1}_n) T_1', (E \;\; F \;\; \mathbf{1}_n) T_2'\right], \\ \Theta_1^* &= \Lambda\Theta^* = (\beta_{11} \; \tau_{11} \; \beta_{(11)}' \; \tau_{(11)}' \; \mu_1 \; \beta_{12} \; \tau_{12} \; \beta_{(12)}' \; \tau_{(12)}' \; \mu_2)', \end{aligned}$$

$$\beta_{(11)}' = (\beta_{21} \; \beta_{31} \; \cdots \; \beta_{b1}).$$

$$\beta_{(12)}' = (\beta_{22} \; \beta_{32} \; \cdots \; \beta_{b2}).$$

$$\tau_{(11)}' = (\tau_{21} \; \tau_{31} \; \cdots \; \tau_{t1}).$$

$$\tau_{(12)}' = (\tau_{22} \; \tau_{32} \; \cdots \; \tau_{t2}).$$

$$\Psi = [\varepsilon_1' \;\; \varepsilon_2']'.$$

Parameter constraints is given in Equation (13).

$$\Omega_1^* \Theta_1^* = 0 \tag{13}$$

with:

$$\Omega_1^* = \begin{bmatrix} G_1^{**} & 0 \\ 0 & G_2^{**} \end{bmatrix} = diag(G_1^{**}, G_2^{**}).$$

$$G_l^{**} = \begin{bmatrix} 1 & 0 & \mathbf{1}_{b-1}' & \mathbf{0}_{t-1}' & 0 \\ 0 & 1 & \mathbf{0}_{b-1}' & \mathbf{1}_{t-1}' & 0 \end{bmatrix}.$$

The third transformation of Equation (12):

$$Y = \Gamma_1^* \Lambda^{*\prime} \Lambda^* \Theta_1^* + \Psi$$

equivalent to Equation (14).

$$Y = \Gamma_2^* \Theta_2^* + \Psi \tag{14}$$

with:

$$\Gamma_2^* = \Gamma_1^* \Lambda^{*\prime}.$$

$$\Theta_2^* = \Lambda^* \Theta_1^* = (\beta_{11}\ \tau_{11}\ \beta_{12}\ \tau_{12}\ \beta'_{(11)}\ \tau'_{(11)}\ \mu_1\ \beta'_{(12)}\ \tau'_{(12)}\ \mu_2)'.$$

$$\Psi = [\varepsilon_1'\ \ \varepsilon_2']'.$$

Equation (15) shows parameter constraints.

$$\Omega_2^* \Theta_2^* = 0 \tag{15}$$

with:

$$\Omega_2^* = (I_4\ \ \Delta).$$

$$\Delta = diag\left[\begin{pmatrix} \mathbf{1}'_{b-1} & \mathbf{0}'_{t-1} & 0 \\ \mathbf{0}'_{b-1} & \mathbf{1}'_{t-1} & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1}'_{b-1} & \mathbf{0}'_{t-1} & 0 \\ \mathbf{0}'_{b-1} & \mathbf{1}'_{t-1} & 0 \end{pmatrix}\right].$$

Next, partition $\Omega_2^* = [\Omega_{21}^*\ \ \Omega_{22}^*]$, $\Gamma_2^* = [\Gamma_{21}^*\ \ \Gamma_{22}^*]$ and $\Theta_2^* = \begin{bmatrix}\Theta_{21}^* \\ \Theta_{22}^*\end{bmatrix}$, so that linear constraints can be written as

$$\Omega_{21}^*\Theta_{21}^* + \Omega_{22}^*\Theta_{22}^* = 0, \quad \text{for } g = 0.$$

because $\Omega_{21}^* = I_4$, so:

$$\Theta_{21}^* + \Omega_{22}^*\Theta_{22}^* = 0 \implies \Theta_{21}^* = -\Omega_{22}^*\Theta_{22}^*.$$

Substituting into Equation (14), we obtain Equation (16).

$$\begin{aligned} Y &= \Gamma_{21}^*\Theta_{21}^* + \Gamma_{22}^*\Theta_{22}^* + \Psi \\ &= \Gamma_{21}^*(-\Omega_{22}^*\Theta_{22}^*) + \Gamma_{22}^*\Theta_{22}^* + \Psi \\ &= (\Gamma_{22}^* - \Gamma_{21}^*\Omega_{22}^*)\Theta_{22}^* + \Psi. \end{aligned}$$

so that the model is without constraints:

$$Y = \Gamma_r \Theta_r + \Psi \tag{16}$$

with:

$$\Gamma_r = \Gamma_{22}^* - \Gamma_{21}^*\Omega_{22}^*.$$

$$\Theta_r = \Theta_{22}^* = (\beta'_{(11)}\ \tau'_{(11)}\ \mu_1\ \beta'_{(12)}\ \tau'_{(12)}\ \mu_2)'.$$

$$\Psi = [\varepsilon_1'\ \ \varepsilon_2']'.$$

Γ_r equivalent to $\Gamma_r = AD$
where

$$A = diag\left([E\ \ F_{(t)}], [E\ \ F_{(t)}]\right).$$

$$D = diag\left([I_{b+t-1}\ \ C], [I_{b+t-1}\ \ C]\right)\Lambda' diag\left(\begin{bmatrix}-\Omega_{22}^* \\ I_{b+t-1}\end{bmatrix}, \begin{bmatrix}-\Omega_{22}^* \\ I_{b+t-1}\end{bmatrix}\right).$$

$$C = \begin{bmatrix} \mathbf{1}_b & \mathbf{1}_b \\ -I_{t-1} & \mathbf{0}_{t-1} \end{bmatrix}.$$

The matrix $F_{(t)}$ is a matrix F without the last column with size $n \times (t-1)$. D is a nonsingular matrix [16], so:

$$\text{rank}(\Gamma_r) = \text{rank}(A) = 2(b+t-1)$$

The design matrix Γ_r has a rank of $2(b+t-1)$, which is equal to the number of parameters in $\boldsymbol{\Theta}_r$. Therefore, Equation (16) is a full-rank model.

3.2 Parameter Estimation

Using Model (16), the Least Squares estimator for $\boldsymbol{\Theta}_r$ is

$$\begin{aligned} \Psi'\Psi &= (Y - \Gamma_r\boldsymbol{\Theta}_r)'(Y - \Gamma_r\boldsymbol{\Theta}_r) \\ &= (Y' - \boldsymbol{\Theta}_r'\Gamma_r')(Y - \Gamma_r\boldsymbol{\Theta}_r) \\ &= Y'Y - Y'\Gamma_r\boldsymbol{\Theta}_r - \boldsymbol{\Theta}_r'\Gamma_r'Y + \boldsymbol{\Theta}_r'\Gamma_r'\Gamma_r\boldsymbol{\Theta}_r \\ &= Y'Y - 2\boldsymbol{\Theta}_r'\Gamma_r'Y + \boldsymbol{\Theta}_r'(\Gamma_r'\Gamma_r)\boldsymbol{\Theta}_r. \end{aligned}$$

Using derivative rules against $\boldsymbol{\Theta}_r$,

$$\begin{aligned} \frac{\partial\Psi'\Psi}{\partial\boldsymbol{\Theta}_r} &= \frac{\partial}{\partial\boldsymbol{\Theta}_r}\left[Y'Y - 2\boldsymbol{\Theta}_r'\Gamma_r'Y + \boldsymbol{\Theta}_r'(\Gamma_r'\Gamma_r)\boldsymbol{\Theta}_r\right] \\ &= -2\Gamma_r'Y + 2(\Gamma_r'\Gamma_r)\boldsymbol{\Theta}_r. \end{aligned}$$

Equating the derivative to zero,

$$\begin{aligned} -2\Gamma_r'Y + 2(\Gamma_r'\Gamma_r)\boldsymbol{\Theta}_r &= 0, \\ -\Gamma_r'Y + (\Gamma_r'\Gamma_r)\boldsymbol{\Theta}_r &= 0, \\ (\Gamma_r'\Gamma_r)\boldsymbol{\Theta}_r &= \Gamma_r'Y, \\ (\Gamma_r'\Gamma_r)^{-1}(\Gamma_r'\Gamma_r)\boldsymbol{\Theta}_r &= (\Gamma_r'\Gamma_r)^{-1}\Gamma_r'Y. \end{aligned}$$

so that the Least Squares estimator is obtained as Equation (17).

$$\hat{\boldsymbol{\Theta}}_r = (\Gamma_r'\Gamma_r)^{-1}\Gamma_r'Y \tag{17}$$

Therefore,

$$E(Y) = \Gamma_r \Theta_r,\ \mathrm{Var}(Y) = \sigma^2 I_{2n},\ E(\hat{\Theta}_r) = \Theta_r,$$

and

$$\mathrm{Var}(\hat{\Theta}_r) = (\Gamma_r' \Gamma_r)^{-1} \sigma^2.$$

The Least Squares estimator for σ^2 [12] is given in Equation (18).

$$\hat{\sigma}^2 = \frac{(Y - \Gamma_r \hat{\Theta}_r)'(Y - \Gamma_r \hat{\Theta}_r)}{n} \tag{18}$$

The variance estimator bias correction is shown in Equation (19).

$$\begin{aligned}
E(\hat{\sigma}^2) &= \frac{1}{n} E\Big[(Y - \Gamma_r \hat{\boldsymbol{\theta}}_r)'(Y - \Gamma_r \hat{\boldsymbol{\theta}}_r)\Big] \\
&= \frac{1}{n} E\Big[\big(Y - \Gamma_r (\Gamma_r'\Gamma_r)^{-1}\Gamma_r' Y\big)' \\
&\qquad \big(Y - \Gamma_r (\Gamma_r'\Gamma_r)^{-1}\Gamma_r' Y\big)\Big] \\
&= \frac{1}{n} E\Big[Y'\big(I - \Gamma_r (\Gamma_r'\Gamma_r)^{-1}\Gamma_r'\big) \\
&\qquad \big(I - \Gamma_r (\Gamma_r'\Gamma_r)^{-1}\Gamma_r'\big) Y\Big] \\
&= \frac{1}{n} E\Big[Y'\big(I - \Gamma_r (\Gamma_r'\Gamma_r)^{-1}\Gamma_r'\big) Y\Big] \\
&= \frac{1}{n} \Big[\mathrm{tr}\left(I - \Gamma_r (\Gamma_r'\Gamma_r)^{-1}\Gamma_r'\right)\sigma^2 \\
&\qquad + (\Gamma_r \boldsymbol{\theta}_r)'\left(I - \Gamma_r (\Gamma_r'\Gamma_r)^{-1}\Gamma_r'\right)\Gamma_r \boldsymbol{\theta}_r\Big] \\
&= \frac{1}{n} \Big[\sigma^2 \Big\{ \mathrm{tr}(I) - \mathrm{tr}\left(\Gamma_r (\Gamma_r'\Gamma_r)^{-1}\Gamma_r'\right)\Big\} \\
&\qquad + \boldsymbol{\theta}_r'\Gamma_r'\Gamma_r\boldsymbol{\theta}_r - \boldsymbol{\theta}_r'\Gamma_r'\Gamma_r(\Gamma_r'\Gamma_r)^{-1}(\Gamma_r'\Gamma_r)\boldsymbol{\theta}_r\Big] \\
&= \frac{1}{n} \Big[\sigma^2 \Big\{ \mathrm{tr}(I) - \mathrm{tr}\left(\Gamma_r (\Gamma_r'\Gamma_r)^{-1}\Gamma_r'\right)\Big\}\Big].
\end{aligned} \tag{19}$$

Consider the rank of the matrix:

$$\begin{aligned}
E(\hat{\sigma}^2) &= \frac{1}{n}\sigma^2 \left[2n - 2(b + t - 1)\right] \\
&= \frac{2n - 2(b + t - 1)}{n}\sigma^2.
\end{aligned}$$

$$E(\hat{\sigma}^2) \neq \sigma^2 \quad \rightarrow \quad \text{bias,}$$

so

$$\hat{\sigma}^2 = \frac{1}{2n - 2(b + t - 1)} Y' \left(I - \Gamma_r \Gamma_r^{-}\right) Y$$

It is shown that the obtained estimator meets the Best Linear Unbiased Estimator (BLUE) criteria, namely [22]:

1. $E(\hat{\Theta}_r) = \Theta_r$, with $\hat{\Theta}_r$ being an unbiased estimator for Θ_r.
2. $\mathrm{Var}(\hat{\Theta}_r) \leq \mathrm{Var}(\Theta_r^*)$, where Θ_r^* is another unbiased estimator of Θ_r.

Proof:
Proof (i) has been proven that

$$E(\hat{\Theta}_r) = \Theta_r.$$

For (ii), suppose Equation (20) is the form of the estimator Θ_r^*,

$$\Theta_r^* = \left[(\Gamma_r'\Gamma_r)^{-1}\Gamma_r' + A\right] Y \tag{20}$$

where A is a $2(b + t - 1) \times 2n$ matrix whose elements are real numbers.

The expectation of the estimator Θ_r^* is

$$\begin{aligned}
E(\Theta_r^*) &= E\left[\left((\Gamma_r'\Gamma_r)^{-1}\Gamma_r' + A\right) Y\right] \\
&= \left[(\Gamma_r'\Gamma_r)^{-1}\Gamma_r' + A\right] E(Y) \\
&= \left[(\Gamma_r'\Gamma_r)^{-1}\Gamma_r' + A\right] \Gamma_r \Theta_r \\
&= \left[I + A\Gamma_r\right] \Theta_r.
\end{aligned}$$

Because Θ_r^* is an unbiased estimator for Θ_r, so $E(\Theta_r^*) = \Theta_r$, then $(I + A\Gamma_r)$ is the identity matrix with $A\Gamma_r = 0$.

The variance of the estimator (Θ_r^*) is

$$\begin{aligned}
\mathrm{Var}(\Theta_r^*) &= \mathrm{Var}\left(\left[(\Gamma_r'\Gamma_r)^{-1}\Gamma_r' + A\right] Y\right) \\
&= \left[(\Gamma_r'\Gamma_r)^{-1}\Gamma_r' + A\right] \mathrm{Var}(Y) \left[(\Gamma_r'\Gamma_r)^{-1}\Gamma_r' + A\right]' \\
&= \left[(\Gamma_r'\Gamma_r)^{-1}\Gamma_r' + A\right] \left[\Gamma_r(\Gamma_r'\Gamma_r)^{-1} + A'\right] \sigma^2 \\
&= \Big[(\Gamma_r'\Gamma_r)^{-1}\Gamma_r'\Gamma_r(\Gamma_r'\Gamma_r)^{-1} + (\Gamma_r'\Gamma_r)^{-1}\Gamma_r' A' \\
&\qquad + A\Gamma_r(\Gamma_r'\Gamma_r)^{-1} + AA'\Big]\sigma^2.
\end{aligned}$$

Because $A\Gamma_r = 0$, then $\Gamma_r' A' = 0$, so that $\mathrm{Var}(\Theta_r^*)$ can be written as

$$\begin{aligned}
\mathrm{Var}(\Theta_r^*) &= \left[(\Gamma_r'\Gamma_r)^{-1} + AA'\right] \sigma^2 \\
&= (\Gamma_r'\Gamma_r)^{-1}\sigma^2 + AA'\sigma^2.
\end{aligned}$$

It is proven that

$$\mathrm{Var}(\hat{\Theta}_r) \leq \mathrm{Var}(\Theta_r^*).$$

3.3 Hypothesis Testing

In the full-rank model as shown in Equation (16), hypothesis testing was carried out with the following hypotheses:

$$H_0 : H\Theta_r = h \qquad \text{and} \qquad H_a : H\Theta_r \neq h$$

with

$$H = \left[H^* \; \mathbf{0}_{(b+t-2)\times 2}\right],$$

$$H^* = \begin{bmatrix} I_{b-1} & 0 & -I_{b-1} & 0 \\ 0 & I_{t-1} & 0 & -I_{t-1} \end{bmatrix}, \quad \text{rank}(H^*) = b+t-2,$$

$$h = \mathbf{0}_{(b+t-2)\times 1}.$$

The testing criteria:
Reject H_0 if

$$F_{\text{hit}} \geq F_{\alpha;(b+t-2),\, n-2(b+t-1)},$$

with

$$F_{\text{hit}} = \left(\frac{\begin{array}{c}(H\hat{\Theta}_r - h)' \left[H(\Gamma_r'\Gamma_r)^{-1}H'\right]^{-1} \\ (H\hat{\Theta}_r - h)\end{array}}{Y'(I - \Gamma_r\Gamma_r^{-})Y} \right) \left(\frac{n - 2(b+t-1)}{b+t-2} \right).$$

The null hypothesis (H_0) states that all block effects and treatment effects in the second RCBD are no different from the effects in the first RCBD, so if H_0 is not rejected, then the entire design can be considered homogeneous and suitable for combined analysis. Conversely, if H_0 is rejected, then the treatment effect and/or block effect between RCBDs are not homogeneous so that design merging cannot be carried out [12].

3.4 Simulation

Simulations were performed using SAS software to obtain parameter vector estimates $\hat{\Theta}_r$ of Model (16) and determine the value of the test power through hypothesis testing with $b = 4$ and $t = 3$, so that the estimated reduced parameter vector is

$$\Theta_r = \beta_{2(1)}, \beta_{3(1)}, \beta_{4(1)}, \tau_{2(1)}, \tau_{3(1)}, \beta_{2(2)}, \beta_{3(2)}, \beta_{4(2)}, \tau_{2(2)}, \tau_{3(2)}, \mu_1, \mu_2.$$

3.4.1 Parameter Estimation

Parameter estimation simulations were performed through 1000 repetitions with parameter vectors

$$\Theta_r = \{-0.5,\ 0.8,\ 1.5,\ -1.2,\ 2.5,\ -0.9,\ 0.4,\ 1.2,\ -0.8,\ 2.0,\ 10,\ 20\}$$

under two error variance conditions, namely $\sigma^2 = 1$ and $\sigma^2 = 9$. The unbiasedness of the estimator is evaluated by averaging all the resulting estimators, as shown in Equation (21).

$$E(\hat{\Theta}_r) = \frac{\hat{\Theta}_{r(1)} + \hat{\Theta}_{r(2)} + \cdots + \hat{\Theta}_{r(1000)}}{1000}. \tag{21}$$

which is used to assess whether $\hat{\Theta}_r$ is an unbiased estimator of the true parameter Θ_r. Figure 1 displays the distribution pattern of Parameter Estimates (β), while Figure 2 shows the distribution pattern of Parameter Estimates (τ).

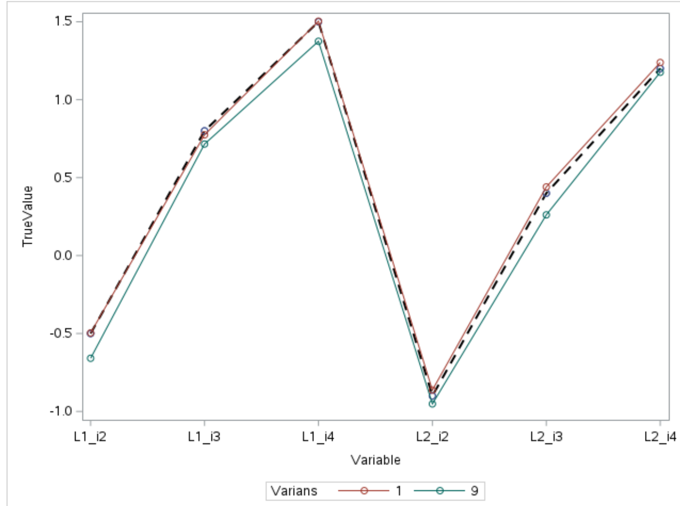

Figure 1. Distribution Pattern of Parameter Estimates (β)

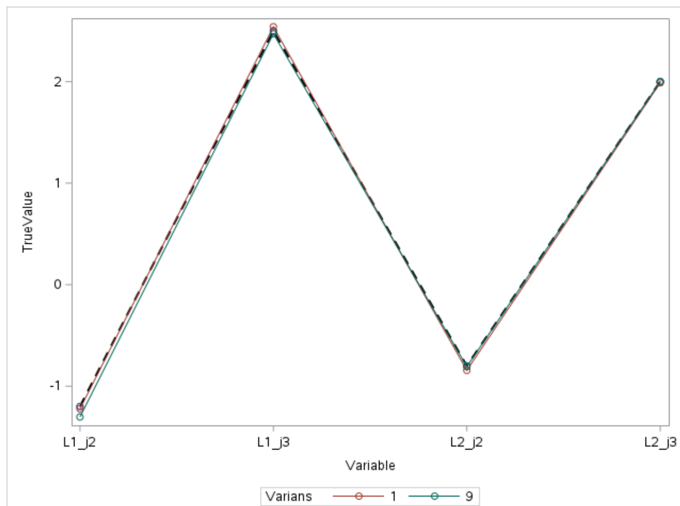

Figure 2. Distribution Pattern of Parameter Estimates (τ)

Table 1 shows that the expected value of the estimator $\hat{\Theta}_r$ in both error variance conditions is close to the actual parameter value Θ_r, so that the resulting estimator is unbiased. However, when the error variance value (σ^2) is increased, the expected value of the estimator tends to experience a larger deviation from the actual parameter value.

The quality of the estimator is evaluated through the Root Mean Square Error (RMSE) value, which represents the precision level and the accuracy of the estimated results. Under conditions $\sigma^2 = 1$ produces RMSE values in the low range, namely around 0.69 to 0.85, which indicates that the variability of the estimator is relatively small and the estimation results are stable approaching the value actually. However, at condition $\sigma^2 = 9$, the RMSE value experienced a significant increase, namely around 2.04 to 2.52. The increase in the RMSE value indicates that the greater variation in the data, the greater the uncertainty in parameter es-

Table 1. Parameter Estimation Simulation Results

σ^2	Θ_r	True Value	$E(\hat{\Theta}_r)$	Var$(\hat{\Theta}_r)$	RMSE
1	$\beta_{2(1)}$	-0.5	-0.4962	0.6640	0.8149
	$\beta_{3(1)}$	0.8	0.7746	0.6509	0.8072
	$\beta_{4(1)}$	1.5	1.5033	0.6919	0.8318
	$\tau_{2(1)}$	-1.2	-1.2233	0.4997	0.7073
	$\tau_{3(1)}$	2.5	2.5411	0.4868	0.6989
	$\beta_{2(2)}$	-0.9	-0.8627	0.6827	0.8271
	$\beta_{3(2)}$	0.4	0.4402	0.7300	0.8553
	$\beta_{4(2)}$	1.2	1.2384	0.6867	0.8296
	$\tau_{2(2)}$	-0.8	-0.8445	0.4792	0.6937
	$\tau_{3(2)}$	2.0	1.9882	0.5158	0.7183
	μ_1	10	9.9907	0.4860	0.6972
	μ_2	20	19.9847	0.5421	0.7364
9	$\beta_{2(1)}$	-0.5	-0.6587	6.0875	2.4724
	$\beta_{3(1)}$	0.8	0.7153	6.3520	2.5217
	$\beta_{4(1)}$	1.5	1.3745	5.6633	2.3830
	$\tau_{2(1)}$	-1.2	-1.3031	4.6508	2.1590
	$\tau_{3(1)}$	2.5	2.4708	4.4846	2.1179
	$\beta_{2(2)}$	-0.9	-0.9514	5.7817	2.4050
	$\beta_{3(2)}$	0.4	0.2611	6.1807	2.4899
	$\beta_{4(2)}$	1.2	1.1757	6.0354	2.4568
	$\tau_{2(2)}$	-0.8	-0.8073	4.1737	2.0429
	$\tau_{3(2)}$	2.0	2.0013	4.5690	2.1375
	μ_1	10	10.1197	4.3681	2.0934
	μ_2	20	20.0864	4.7000	2.1696

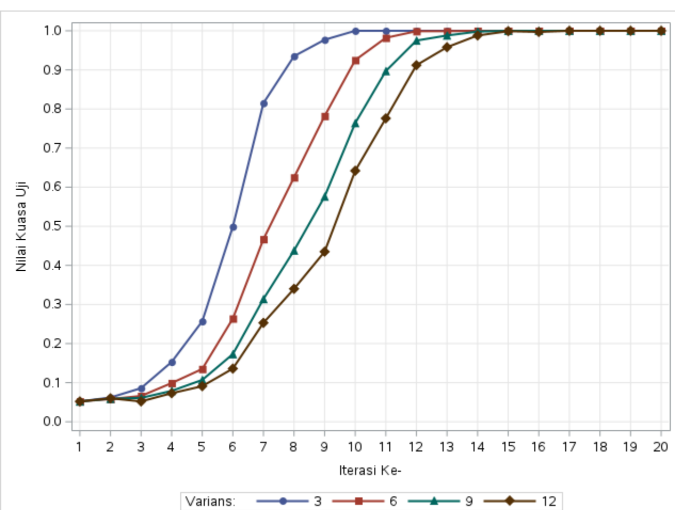

Figure 3. Test Power Value Graph

timates. Although the estimator remains unbiased, its precision decreases as the error distribution increases.

3.4.2 Test Power

Hypothesis testing was conducted using $\alpha = 0.05$ and 1000 randomly generated data sets. The hypotheses tested were

$$H_0 : \begin{cases} \beta_{i(1)} = \beta_{i(2)}, & \forall i = 2, 3, 4, \\ \tau_{j(1)} = \tau_{j(2)}, & \forall j = 2, 3, \end{cases}$$

$$H_a : \text{ at least one parameter is different.}$$

Testing was conducted under several different parameter conditions and error ranges. Figure 3 displays the pattern of changes in test power values.

In the condition $\sigma^2 = 3$, the power value reaches one earlier as the iterations increase, indicating that the testing procedure has high detection capability under conditions of low error variance. The power value for conditions $\sigma^2 = 6$, $\sigma^2 = 9$ and $\sigma^2 = 12$ decreases slightly, which is caused by the increase in variance, which increases the error in the data and reduces the chance of incorrectly rejecting H_0. However, the power value increases

Table 2. Test Power Values

Resampling	Parameter $\boldsymbol{\Theta}_r$	σ^2			
		1	6	9	12
1	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	0.0520	0.0520	0.0520	0.0520
2	$(0.1, -0.1, 0.15, -0.15, 0.2, -0.2, 0.25, -0.25, 0.4, 0.5)$	0.0620	0.0580	0.0590	0.0600
3	$(0.2, -0.2, 0.3, -0.3, 0.4, -0.4, 0.5, -0.5, 0.8, 1.0)$	0.0860	0.0660	0.0610	0.0520
4	$(0.35, -0.35, 0.5, -0.5, 0.65, -0.65, 0.8, -0.8, 1.15, 1.4)$	0.1530	0.0990	0.0790	0.0730
5	$(0.5, -0.5, 0.7, -0.7, 0.9, -0.9, 1.1, -1.1, 1.5, 1.8)$	0.2570	0.2570	0.1070	0.0910
6	$(0.75, -0.75, 1.05, -1.05, 1.35, -1.35, 1.65, -1.65, 1.95, 2.2)$	0.4990	0.2640	0.1730	0.1360
7	$(1.0, -1.0, 1.4, -1.4, 1.8, -1.8, 2.2, -2.2, 2.4, 2.6)$	0.8150	0.4670	0.3140	0.2530
8	$(1.5, -1.5, 1.8, -1.8, 2.1, -2.1, 2.3, -2.3, 2.6, 2.8)$	0.9350	0.6250	0.4380	0.3400
9	$(2.0, -2.0, 2.2, -2.2, 2.4, -2.4, 2.4, -2.4, 2.8, 3.0)$	0.9770	0.7820	0.5760	0.4350
10	$(2.5, -2.5, 2.7, -2.7, 2.9, -2.9, 3.0, -3.0, 3.3, 3.5)$	1.0000	0.9250	0.7640	0.6420
11	$(3.0, -3.0, 3.2, -3.2, 3.4, -3.4, 3.6, -3.6, 3.8, 4.0)$	1.0000	0.9820	0.8970	0.7760
12	$(3.5, -3.5, 3.7, -3.7, 3.8, -3.8, 4.0, -4.0, 4.1, 4.3)$	1.0000	0.9990	0.9750	0.9120
13	$(4.0, -4.0, 4.2, -4.2, 4.2, -4.2, 4.4, -4.4, 4.4, 4.6)$	1.0000	0.9990	0.9880	0.9580
14	$(4.5, -4.5, 4.7, -4.7, 4.7, -4.8, 4.9, -4.9, 5.0, 5.2)$	1.0000	1.0000	0.9980	0.9880
15	$(5.0, -5.0, 5.2, -5.2, 5.2, -5.4, 5.4, -5.4, 5.6, 5.8)$	1.0000	1.0000	1.0000	0.9990
16	$(5.2, -5.2, 5.3, -5.4, 5.4, -5.6, 5.6, -5.7, 5.8, 5.9)$	1.0000	1.0000	1.0000	0.9970
17	$(5.4, -5.4, 5.4, -5.6, 5.6, -5.8, 5.8, -6.0, 6.0, 6.0)$	1.0000	1.0000	1.0000	1.0000
18	$(5.2, -5.2, 5.9, -6.0, 6.2, -6.3, 6.7, -7.4, 7.4, 7.5)$	1.0000	1.0000	1.0000	1.0000
19	$(5.0, -5.0, 6.4, -6.4, 6.8, -6.8, 7.6, -8.8, 8.8, 9.0)$	1.0000	1.0000	1.0000	1.0000
20	$(4.6, -4.6, 7.4, -7.2, 8.0, -7.8, 9.4, -10.2, 10.2, 12.0)$	1.0000	1.0000	1.0000	1.0000

again and even reaches one in subsequent iterations with a larger parameter distribution. This increase again indicates that the testing procedure maintains its sensitivity to differences in the parameters being tested. Table 2 presents the power value for each condition.

4. CONCLUSIONS

Based on the results and discussion, it can be concluded that: The research results prove that the Model Reduction Method (MRM) is effective for transforming a non-full rank model containing constraints into a full rank model without constraints in a combination of two Randomized Complete Block Design (RCBD) models. Parameter estimation in the full rank model produces an estimator that is unbiased and has minimum variance, in accordance with the Best Linear Unbiased Estimator (BLUE) criteria. The results of the hypothesis testing show that the model obtained has good power of test under low error variance conditions, and continues to show consistent performance reaching a value of one under all error variance conditions as the distribution of the parameter increases.

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