



Research Paper

On Generalized Semiderivations and Semimultipliers in Semiprime Ideals

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Abstract

In this study, we examine the behavior of a ring A when it admits a generalized semiderivation Φ associated with a semiderivation Ω and a mapping σ , as well as semimultipliers T within its semiprime ideals P . More specifically, the present work investigates differential identities in the semiprime ideal of an arbitrary ring using P -commuting semiderivations.

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1. INTRODUCTION

Throughout this study consistently denote an associative ring A with $Z(A)$ as its center. Usually, the symbols $[r, s]$ and $r \circ s$ denote the commutator $rs - sr$ and the anticommutator $rs + sr$, respectively. Also, we shall frequently refer to the identities listed below that are connected to commutators and anticommutators without naming them individually every time.

$$\begin{aligned} [v, \omega t] &= \omega[v, t] + [v, \omega]t \quad \text{and} \\ [v\omega, t] &= v[\omega, t] + [v, t]\omega, \end{aligned} \quad (1)$$

$$(v\omega \circ t) = v(\omega \circ t) - [v, t]\omega = (v \circ t)\omega + v[\omega, t], \quad (2)$$

$$v \circ (\omega t) = (v \circ \omega)t - \omega[v, t] = \omega(v \circ t) + [v, \omega]t. \quad (3)$$

Let us recall that an ideal P of A is defined as prime if $P \neq A$ and for all $r, s \in A$, $rAs \subseteq P$ implies that $r \in P$ or $s \in P$. So if the ideal (0) is prime, then A is referred to as prime ring. P is a semiprime ideal if $P \neq A$ and for all $r \in A$, $rAr \subseteq P$.

C_P implies that $r \in P$ and A is a semiprime ring if $P = 0$ is a semiprime. Let P be a nonempty subset of A . A mapping Φ from A to A is called commuting on P if $[\Phi(v), v] = 0$, for all $v \in P$. Generalisations of this definition include: A mapping Φ from A to A is called a P -commuting map on P if $[\Phi(v), v] \in P$ for all $v \in P$ and some $P \subset A$. In particular, if $P = 0$, then Φ is called a commuting map on A if $[\Phi(v), v] = 0$. Note that every commuting map is a P -commuting map (put $0 = P$). But the converse is not true in general.

A derivation D is an additive mapping such that $D(v\omega) = D(v)\omega + vD(\omega)$ holds for all $v, \omega \in A$. A map $F : A \rightarrow A$ is called a generalized derivation of A if $F(v\omega) = F(v)\omega + vD(\omega)$ holds for all $v, \omega \in A$. An additive map $T : A \rightarrow A$ is called a left (resp. right) multiplier if $T(v\omega) = T(v)\omega$ (resp. $T(v\omega) = vT(\omega)$) holds for every $v, \omega \in A$. If T is both a left and a right multiplier of A , then it is called a multiplier of A . An ample of work has been done on left (resp. right) centralizers in prime and semiprime rings during the last few decades. See ([1], [2], [3], [4], [5]).

In [6], Bergen defined the concept of a semiderivation of a ring A as follows: an additive map Ω of A into itself is called a semiderivation if there exists a function $\sigma : A \rightarrow A$ such that

$$\Omega(v\omega) = \Omega(v)\sigma(\omega) + v\Omega(\omega) = \Omega(v)\omega + \sigma(v)\Omega(\omega),$$

$$\Omega(\sigma(v)) = \sigma(\Omega(v))$$

for all $v, \omega \in A$. In case σ is the identity map of A , Ω is a derivation.

Let Ω be a semiderivation of A , associated with an endomorphism σ . The additive map Φ on A is a generalized semiderivation of A associated with semiderivation if

$$\Phi(v\omega) = \Phi(v)\omega + \sigma(v)\Omega(\omega) = \Phi(v)\sigma(\omega) + v\Omega(\omega)$$

and

$$\Phi(\sigma(v)) = \sigma(\Phi(v)),$$

for all $v, \omega \in A$.

In 2018, K. H. Kim [7] gave the definition of a semimultiplier, which was influenced by centralizers like α -centralizers. An additive mapping $\Phi : A \rightarrow A$ is called a semimultiplier associated with a surjective function $\sigma : A \rightarrow A$ if

$$(i) T(v\omega) = T(v)\sigma(\omega) = \sigma(v)T(\omega), \quad (ii) T(\sigma(v)) = \sigma(T(v)),$$

for every $v, \omega \in A$.

In [8], Dhara showed that a nonzero generalized derivation F of a semiprime ring A associated with a derivation D of nonzero ideal I such that

$$F(v\omega) = F(v)F(\omega) \quad \text{or} \quad F(v\omega) = F(\omega)F(v)$$

for all $v, \omega \in I$, then $D(I) = 0$ or A contains a nonzero central ideal. In particular, if a prime ring A is commutative and F is left multiplier of A . Albas [9] proved that if a prime ring A admits a nonzero generalized derivation F with associated derivation D , then either F is an identity map of the ring A or A is commutative.

In several papers all of these identities are also examined in certain suitable subsets of prime and semiprime rings. See [10], [9], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20] where further references can be found. In 2020, to extend the theory of derivations rings, which has been studied for years, Almahdi et al set out to examine the derivations of a random ring A satisfying some P -valued conditions where P is the prime ideal of A . As an important development in this work, they stated the theorem known as Posner's Second Theorem as follows: Let A be a ring, P is a prime ideal of A and D a derivation of A . If $[[D(v), v], \omega] \in P$ for all $v, \omega \in A$ then $\Omega(A) \subseteq P$ or S/P is commutative. Recently, Kim [15] proved $*$ -semimultipliers in the prime rings with involution and showed that A is commutative.

Motivated by these developments, the present work studies generalized semiderivations together with semimultipliers in the setting of semiprime ideals. Unlike the existing literature, we investigate several differential identities involving commutators and anticommutators and establish sufficient conditions under which the associated automorphism σ is P -commuting on a nonzero ideal. Thus, the obtained results extend and complement earlier commutativity results by incorporating generalized semiderivations and semimultipliers into a unified framework.

2. MAIN RESULTS

The following lemma plays a crucial role in establishing the main results of this paper. It is repeatedly used in the proofs of the theorems to deduce certain commutator expressions belonging to the semiprime ideal P .

Lemma 2.1 ([13], Lemma 1). Suppose that A be a ring with P a semiprime ideal of A such that $P \subset \Lambda$, Λ an ideal of A , and $a \in \Lambda$ such that $axa \in P$ for all $x \in \Lambda$, then $a \in P$.

Theorem 2.2. Let Λ be a nonzero ideal, P a semiprime ideal of a ring and σ be an automorphism on A . Suppose that A admits a nonzero generalized semiderivation Φ and a semimultiplier T associated with σ . If $\Phi[v, \omega] \pm T[v, \omega] \in P$ for all $v, \omega \in \Lambda$ then σ is P -Commuting on Λ .

Proof. Suppose that

$$\Phi[v, \omega] \pm T[v, \omega] \in P \quad \text{for all } v, \omega \in \Lambda. \quad (2.2.1)$$

Replace ω by ωv in (2.2.1), we obtain

$$(\Phi[v, \omega] \pm T[v, \omega])\sigma(v) + [v, \omega]\Omega(v) \in P \quad \text{for all } v, \omega \in \Lambda. \quad (2.2.2)$$

Using (2.2.1), we get

$$[v, \omega]\Omega(v) \in P \quad \text{for all } v, \omega \in \Lambda. \quad (2.2.2)$$

Replacing ω by $r\omega$, $r \in A$ and using (2.2.2), we find that

$$[v, r]\omega\Omega(v) \in P \quad \text{for all } v, \omega \in \Lambda, r \in A. \quad (2.2.3)$$

Replace ω by ωv in (2.2.3), we obtain

$$[v, r]\omega v\Omega(v) \in P \quad \text{for all } v, \omega \in \Lambda, r \in A. \quad (2.2.4)$$

Multiplying (2.2.3) by v from the right, we get

$$[v, r]\omega\Omega(v)v \in P \quad \text{for all } v, \omega \in \Lambda, r \in A. \quad (2.2.5)$$

Subtracting expressions (2.2.4) and (2.2.5) we get that

$$[v, r]\omega[\Omega(v), v] \in P \quad \text{for all } v, \omega \in \Lambda, r \in A. \quad (2.2.6)$$

Writing $\Omega(v)$ instead of r in last expression, we get

$$[\delta(v), v]\omega[\sigma(v), v] \in P \quad \text{for all } v, \omega \in \Lambda. \quad (2.2.7)$$

It follows that

$$[\sigma(v), v]\Lambda[\sigma(v), v] \in P \quad \text{for all } v, \omega \in \Lambda. \quad (2.2.8)$$

By Lemma 2.1, we arrive at $[\sigma(v), v] \in P$ for all $v \in \Lambda$. Hence σ is P -commuting on Λ .

Theorem 2.3. Let Λ be a nonzero ideal, P a semiprime ideal of a ring and σ be an automorphism on A . Suppose that A admits a nonzero generalized semiderivation Φ and a semimultiplier T associated with σ . If $\Phi(v \circ \omega) \pm T(v \circ \omega) \in P$ for all $v, \omega \in \Lambda$ then σ is P -Commuting on Λ .

Proof. By the Hypothesis, we have

$$\Phi(v \circ \omega) \pm T(v \circ \omega) \in P \quad \text{for all } v, \omega \in \Lambda. \quad (2.3.1)$$

Replace ω by ωv in (2.3.1), we obtain

$$\Phi(v \circ \omega) \pm T(v \circ \omega)\sigma(v) + (v \circ \omega)\Omega(v) \in P \quad \text{for all } v, \omega \in \Lambda. \quad (4)$$

Using (2.3.1), we get

$$(v \circ \omega)\sigma(v) \in P \text{ for all } v, \omega \in \Lambda. \quad (2.3.2)$$

Substituting $r\omega$ for ω , $r \in A$ and using (2.3.2), we find that

$$[v, r]\omega\sigma(v) \in P \text{ for all } v, \omega \in \Lambda, r \in A. \quad (2.3.3)$$

Using a method similar to Theorem 2.2 of (2.2.3), we obtain our conclusion. Following results can analogously be obtained.

Theorem 2.4. Let Λ be a nonzero ideal, P a semiprime ideal of a ring and σ be an automorphism on A . Suppose that A admits a nonzero generalized semiderivation Φ and a semimultiplier T associated with σ . If $[\Phi[v, \omega] \pm T(v\omega)] \in P$ for all $v, \omega \in \Lambda$, then σ is P -Commuting on Λ .

Proof. The proof follows the same argument as in Theorem 2.2. Replacing the hypothesis of Theorem 2.2 by

$$\Phi[v, \omega] \pm T(v\omega) \in P \quad \text{for all } v, \omega \in \Lambda.$$

The same argument together with Lemma 2.1 yields $[\sigma(v), v] \in P$ for all $v \in \Lambda$. Therefore, σ is P -commuting on Λ . $\square$

Theorem 2.5. Let Λ be a nonzero ideal, P a semiprime ideal of a ring and σ be an automorphism on A . Suppose that A admits a nonzero generalized semiderivation Φ and a semimultiplier T associated with σ . If $\Phi(v \circ \omega) \pm T(v\omega) \in P$ for all $v, \omega \in \Lambda$, then σ is P -Commuting on Λ .

Proof. The proof is analogous to that of Theorem 2.3. Using

$$\Phi(v \circ \omega) \pm T(v\omega) \in P \quad \text{for all } v, \omega \in \Lambda.$$

In place of the corresponding hypothesis and following the same substitutions, Lemma 2.1 gives $[\sigma(v), v] \in P$ for all $v \in \Lambda$. Therefore, σ is P -commuting on Λ . $\square$

Theorem 2.6. Let Λ be a nonzero ideal, P a semiprime ideal of a ring and σ be an automorphism on A . Suppose that A admits a nonzero generalized semiderivation Φ and a semimultiplier T associated with σ . If $\Phi[v, \omega] \pm T(v)T(\omega) \in P$ for all $v, \omega \in \Lambda$, then σ is P -Commuting on Λ .

Proof. Following the proof of Theorem 2.2 and replacing its hypothesis by

$$\Phi[v, \omega] \pm T(v)T(\omega) \in P \quad \text{for all } v, \omega \in \Lambda.$$

The same argument together with Lemma 2.1 yields $[\sigma(v), v] \in P$ for all $v \in \Lambda$. Therefore, σ is P -commuting on Λ . $\square$

Theorem 2.7. Let Λ be a nonzero ideal, P a semiprime ideal of a ring and σ be an automorphism on A . Suppose that A admits a nonzero generalized semiderivation Φ and a semimultiplier T associated with σ . If $\Phi(v \circ \omega) \pm T(v)T(\omega) \in P$ for all $v, \omega \in \Lambda$, then σ is P -Commuting on Λ .

Proof. The Proofs of Theorems 2.7 follow the same line of argument as that of Theorem 2.2. In each case, the corresponding

hypothesis replaces equation (2.2.1), and the subsequent substitutions and application of Lemma 2.1 yields $[\sigma(v), v] \in P$ for all $v \in \Lambda$. Therefore, σ is P -commuting on Λ . $\square$

Corollary 2.8. Let P a semiprime ideal of a ring A admitting a generalized semiderivation Φ associated with a semiderivation Ω and a map σ . If A satisfies any one of the following identities: (i). $\Phi[v, \omega] \pm T[v, \omega] \in P$ (ii). $\Phi(v \circ \omega) \pm T(v\omega) \in P$ (iii). $\Phi[v, \omega] \pm T(v\omega) \in P$ (iv). $\Phi(v \circ \omega) \pm T(v\omega) \in P$ (v). $\Phi[v, \omega] \pm T(v)T(\omega) \in P$ (vi). $\Phi(v \circ \omega) \pm T(v)T(\omega) \in P$ for all $v, \omega \in A$, then σ is P -Commuting on Λ .

Proof. The conclusion follows immediately from Theorems 2.2–2.7, since each of the identities (i)–(vi) satisfies the corresponding hypotheses of these theorems. Therefore, in every case, σ is P -commuting on Λ . $\square$

3. EXAMPLE

Example 1. Consider S be set of integers and

$$A = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \mid \alpha, \beta, \gamma \in S \right\}.$$

Define maps Φ, Ω, σ and $T : A \rightarrow A$ by

$$\Phi \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & \alpha\beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Omega \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \sigma \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

$$T \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

It is readily verified that T is a semimultiplier associated with the automorphism σ , and that Φ is a generalized semiderivation associated with the semiderivation Ω . Moreover, the identity $\Phi[v, \omega] \pm T(v)T(\omega) \in P$ for all $v, \omega \in A$ is satisfied. Hence, all the hypotheses of Theorem 2.6 are fulfilled. Therefore, by Theorem 2.6, σ is P -commuting on A . Thus, the above example illustrates the applicability of Theorem 2.6.

Example 2. Let $A = M_2(\mathbb{Z})$ and $P = \{0\}$. Define maps Φ, Ω, σ and $T : A \rightarrow A$ by $\Phi(X) = 0$, $\Omega(X) = 0$, $\sigma(X) = X$, $T(X) = 0$ for every $X \in A$. Clearly, σ is an automorphism of A , T is a semimultiplier associated with σ , and Φ is a generalized semiderivation associated with the semiderivation Ω . Thus, the hypotheses of Theorems 2.4, 2.5, and 2.7 are satisfied. Consequently, σ is P -commuting on A by Theorems 2.4, 2.5, and 2.7, respectively.

4. CONCLUSIONS

In this paper, we investigated differential identities involving generalized semiderivations associated with semiderivations and semimultipliers in semiprime ideals of arbitrary rings. By considering various identities involving commutators, anticommutators, and semimultipliers, we established sufficient conditions under which the associated automorphism σ is P -commuting on the nonzero ideal Λ . An illustrative example was presented to demonstrate the applicability of the obtained results. These findings extend the study of generalized semiderivations and semimultipliers in semiprime ideals and provide further insight into the structural properties of noncommutative rings.

The techniques developed in this work may be extended to broader classes of algebraic structures, including prime rings, rings with involution, Banach algebras, and other derivation-like mappings. Future research may also investigate analogous differential identities under weaker assumptions or in the presence of different classes of multipliers and endomorphisms, leading to a deeper understanding of the commutativity and structural properties of noncommutative rings.

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