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Research Paper

DSm Fuzzy Vector Space, DSm Neutrosophic Vector Space, and DSm Plithogenic Vector Space

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Abstract

A wide range of formalisms has been developed to represent uncertainty across disciplines, including Fuzzy Sets, Rough Sets, Intuitionistic Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets, among others that continue to be active areas of research. A DSm Vector Space is defined as a linear space over refined DSm labels, where addition and scalar multiplication satisfy the classical axioms. In this paper, we newly introduce the extensions of DSm Vector Space using Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets, namely the DSm Fuzzy Vector Space, the DSm Neutrosophic Vector Space, and the DSm Plithogenic Vector Space, and investigate their structural properties.

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1. INTRODUCTION

When modeling real-world phenomena, uncertainty cannot be neglected. To capture such uncertainty, a variety of set-theoretic frameworks have been introduced, including fuzzy sets [1], intuitionistic fuzzy sets [2], hyperfuzzy sets [3], neutrosophic sets [4, 5], hesitant fuzzy sets [6], bipolar neutrosophic sets [7, 8], Functorial sets [9, 10], and plithogenic sets [11]. These structures have been applied and exploited in many contexts, particularly in decision science.

A (classical) vector space is a set equipped with vector addition and scalar multiplication that satisfy the usual axioms of associativity, distributivity, and the existence of identity elements and additive inverses. Building on this, the notion of a vector space has been extended by employing fuzzy sets and neutrosophic sets, leading to fuzzy vector spaces [12, 13] and neutrosophic vector spaces [14, 15], among others. A DSm vector space is defined as a linear space over refined DSm labels, where addition and scalar multiplication satisfy the classical axioms [16, 17].

Research on DSm vector spaces formalizes linear operations on refined labels, unifying evidence theory and linear algebra for uncertainty modeling in decision-making, fusion, and learning. Unlike classical vector spaces over numerical fields, DSm vector spaces use refined DSm labels as scalars, encoding belief and conflict, and thereby enable linear computation together with evidence fusion for uncertain, conflicting, and imprecise data.

In view of the above, research on Fuzzy Sets, Neutrosophic Sets, Plithogenic Sets, and DSm vector spaces is important; however, studies that fuse Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets with DSm Vector Spaces have not yet been carried out extensively. In this paper, we newly introduce the extensions of DSm Vector Space using Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets, namely the DSm Fuzzy Vector Space, the DSm Neutrosophic Vector Space, and the DSm Plithogenic Vector Space, and investigate their structural properties.

2. PRELIMINERIES

This section assembles the basic terminology and notation used throughout the paper. Unless stated otherwise, we work in the finite setting. By convention, the empty set is regarded as an element of every set.

2.1 Fuzzy Set and Neutrosophic Set

A fuzzy set assigns to each element a grade within the interval $[0, 1]$, representing uncertainty through degrees rather than by a strict binary division [18]. Extended frameworks have been introduced, including Intuitionistic Fuzzy Sets [2, 19], Single-Valued Neutrosophic Sets [20, 21], and Hesitant Fuzzy Sets [6, 22]. Below, we recall the relevant notions along with their extensions.

Definition 2.1 (Fuzzy Set) *Let Y be a nonempty universe. A* fuzzy set *on Y is a mapping*

$$\tau : Y \to [0, 1].$$

A fuzzy relation *on Y is a fuzzy subset*

$$\delta : Y \times Y \to [0, 1].$$

We say that δ is a fuzzy relation on τ if, for all $y, z \in Y$,

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\}.$$

Definition 2.2 (Fuzzy Set: on-demand driver availability)

$$Let\, Y = \{D_1, D_2, D_3\}$$

be drivers in a ride-hailing app. Define the fuzzy set

$$\tau : Y \to [0, 1]$$

measuring "available to pick up within 5 minutes" by

$$\tau(D_1) = 0.90, \qquad \tau(D_2) = 0.60, \qquad \tau(D_3) = 0.30.$$

Let

$$s : Y \times Y \to [0, 1]$$

encode a (symmetric) co-location similarity based on current proximity to the rider (with $s(y, y) = 1$), e.g.,

$$s(D_1, D_2) = 0.80, \qquad s(D_1, D_3) = 0.40, \qquad s(D_2, D_3) = 0.50.$$

Define the fuzzy relation

$$\delta : Y \times Y \to [0, 1]$$

by

$$\delta(y, z) := s(y, z) \cdot \min\{\tau(y), \tau(z)\}.$$

Then

$$\delta(D_1, D_2) = 0.80 \cdot \min\{0.90, 0.60\} = 0.80 \cdot 0.60 = 0.48 \leq 0.60,$$

$$\delta(D_1, D_3) = 0.40 \cdot \min\{0.90, 0.30\} = 0.40 \cdot 0.30 = 0.12 \leq 0.30,$$

$$\delta(D_2, D_3) = 0.50 \cdot \min\{0.60, 0.30\} = 0.50 \cdot 0.30 = 0.15 \leq 0.30.$$

Hence,

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\}$$

for all $y, z \in Y$. Therefore, δ is a fuzzy relation on τ.

Neutrosophic sets extend fuzzy sets by explicitly incorporating *indeterminacy*, thus covering statements that are neither entirely true nor entirely false, and offering a flexible representation of ambiguity [4, 20, 21]. As extensions of neutrosophic sets, concepts such as the Double-Valued Neutrosophic Set [22, 23, 24, 25, 26, 27] and the Bipolar Neutrosophic Set [28, 29] have been introduced. Their formal specification is given below.

Definition 2.3 (Neutrosophic Set) *Let X be a nonempty set. A* Neutrosophic Set (NS) *A on X is determined by three functions*

$$T_A : X \to [0, 1], \qquad I_A : X \to [0, 1], \qquad F_A : X \to [0, 1],$$

where, for each $x \in X$, $T_A(x)$, $I_A(x)$, and $F_A(x)$ denote the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 2.1 (Neutrosophic Set: email spam assessment) *Let*

$$X = \{E_1, E_2, E_3\}$$

be emails and let A denote the statement "is spam". Assign to each email a neutrosophic triplet

$$(T_A, I_A, F_A) \in [0, 1]^3$$

representing degrees of truth (spam), indeterminacy, and falsity (not spam), respectively.

x	$T_A(x)$	$I_A(x)$	$F_A(x)$
E_1	0.85	0.05	0.10
E_2	0.40	0.45	0.35
E_3	0.10	0.20	0.90

All components lie in $[0, 1]$, and

$$T_A + I_A + F_A \in \{1.00, 1.20, 1.20\} \leq 3$$

for the three emails.

Interpretation: E_1 is very likely spam with low uncertainty; E_2 has mixed indicators (high indeterminacy due to conflicting signals); E_3 is strongly assessed as not spam.

2.2 Plithogenic Set

A plithogenic set [11, 30, 31, 32, 33, 34, 35] represents elements via membership driven by explicit attributes together with a contradiction mapping between attribute values.

Definition 2.4 (Plithogenic Set) *Let S be a universe and $P \subseteq S$ a nonempty subset. A* plithogenic set *is a 5-tuple*

$$PS = (P, v, P_v, pdf, pCF),$$

with the following ingredients:

- *v - a chosen attribute;*

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- P_v *- the value domain of* v*;*
- $pdf\ :\ P \times P_v \to [0,1]^s$ *- the degree of appurtenance (DAF);*
- $pCF\ :\ P_v \times P_v \to [0,1]^t$ *- the degree of contradiction (DCF).*

For every $a, b \in P_v$, the DCF satisfies

$$\text{reflexivity: } pCF(a,a) = 0,$$

$$\text{symmetry: } pCF(a,b) = pCF(b,a).$$

Here $s, t \in \mathbb{N}$ are, respectively, the appurtenance and contradiction dimensions.

Example 2.2 ***(University admissions- plithogenic intersection of two committees)***
Let S *be the set of applicants and*

$$P = \{C1\} \subseteq S$$

the candidate under review. Choose the attribute

$$v = \text{"evidence type"}$$

with value domain

$$P_v = \{test,\ recommendation,\ portfolio\},$$

and dominant value

$$v^\star = test.$$

Take scalar DAF ($s = 1$) *and use the product* t*-norm*

$$u \otimes v = uv$$

and the probabilistic s*-norm*

$$u \oplus v = u + v - uv.$$

Degree of contradiction (DCF). *Assume* pCF *is symmetric with*

$$\begin{aligned} pCF(test, test) &= 0, \\ pCF(test, recommendation) &= 0.3, \\ pCF(test, portfolio) &= 0.6, \\ pCF(recommendation, recommendation) &= 0, \\ pCF(recommendation, portfolio) &= 0.4, \\ pCF(portfolio, portfolio) &= 0. \end{aligned}$$

Write

$$c(a) := pCF(a, v^\star),$$

so

$$c(test) = 0, \qquad c(recommendation) = 0.3, \qquad c(portfolio) = 0.6.$$

Committee DAFs. Two committees A, B provide appurtenances for candidate $C1$:

$$pdf_A(C1, \text{test}) = 0.85, \qquad pdf_A(C1, \text{recommendation}) = 0.70,$$
$$pdf_A(C1, \text{portfolio}) = 0.40,$$
$$pdf_B(C1, \text{test}) = 0.80, \qquad pdf_B(C1, \text{recommendation}) = 0.55,$$
$$pdf_B(C1, \text{portfolio}) = 0.60.$$

Plithogenic intersection.
Using the contradiction–weighted mixer

$$u\ {}_{1-c(a)}\!\wedge\ v := (1 - c(a))(u \otimes v) + c(a)(u \oplus v),$$

we obtain (valuewise for $a \in P_v$):

$$\begin{aligned} d_{A\cap B}(C1, \text{test}) &= (1-0)(0.85 \cdot 0.80) + 0(0.85 \oplus 0.80) \\ &= 0.68, \end{aligned}$$

$$\begin{aligned} d_{A\cap B}(C1, \text{recommendation}) &= (1-0.3)(0.70 \cdot 0.55) \\ &\quad + 0.3(0.70 \oplus 0.55) \\ &= 0.7 \cdot 0.385 + 0.3 \cdot 0.865 \\ &= 0.529, \end{aligned}$$

$$\begin{aligned} d_{A\cap B}(C1, \text{portfolio}) &= (1-0.6)(0.40 \cdot 0.60) + 0.6(0.40 \oplus 0.60) \\ &= 0.4 \cdot 0.24 + 0.6 \cdot 0.76 \\ &= 0.552. \end{aligned}$$

Optional scalarization. With weights

$$w(\text{test}) = 0.5, \qquad w(\text{recommendation}) = 0.3,$$
$$w(\text{portfolio}) = 0.2,$$

$$\mu^\star_{A\cap B}(C1) = 1 - \prod_{a \in P_v} (1 - w(a)\, d_{A\cap B}(C1, a)) =$$

$$1 - (1 - 0.34)(1 - 0.1587)(1 - 0.1104) \approx 0.506.$$

This yields a single admission score while keeping DCF-driven mixing at the attribute level.

Example 2.3 ***(Medical treatment adoption-plithogenic union of two evidence sources)*** *Let* S *be candidate therapies and*

$$P = \{T\} \subseteq S$$

be therapy. Choose

$$v = \text{"evaluation criterion"}$$

with

$$P_v = \{efficacy,\ safety,\ affordability\}, \qquad v^\star = efficacy.$$

Use $s = 1$*, the product* t*-norm, and probabilistic* s*-norm.*

[0]In the literature, DAF is modeled in several equivalent ways (e.g., powerset-valued or vector-valued). We adopt the standard $[0,1]^s$ form; see [35].

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Degree of contradiction (DCF). Assume the symmetric map pCF satisfies

$$c(\text{efficacy}) = 0, \qquad c(\text{safety}) = 0.2, \qquad c(\text{affordability}) = 0.7,$$

reflecting that affordability often contradicts efficacy more than safety does.

Two evidence sources. A randomized clinical trial (A) and a real–world evidence study (B) provide DAFs:

$$pdf_A(T, \text{efficacy}) = 0.75, \qquad pdf_A(T, \text{safety}) = 0.65,$$

$$pdf_A(T, \text{affordability}) = 0.30,$$

$$pdf_B(T, \text{efficacy}) = 0.80, \qquad pdf_B(T, \text{safety}) = 0.55,$$

$$pdf_B(T, \text{affordability}) = 0.45.$$

Plithogenic union. Adopting a liberal decision rule, use

$$d_{A \cup B}(T, a) := (1 - c(a))(pdf_A \oplus pdf_B)$$

$$+ c(a)(pdf_A \otimes pdf_B).$$

Then

$$\begin{aligned} d_{A\cup B}(T, \text{efficacy}) &= (1-0)(0.75 \oplus 0.80) + 0(0.75 \cdot 0.80) \\ &= 0.75 + 0.80 - 0.75 \cdot 0.80 \\ &= 0.95, \end{aligned}$$

$$\begin{aligned} d_{A\cup B}(T, \text{safety}) &= (1-0.2)(0.65 \oplus 0.55) + 0.2(0.65 \cdot 0.55) \\ &= 0.8 \cdot 0.8425 + 0.2 \cdot 0.3575 \\ &= 0.7455, \end{aligned}$$

$$\begin{aligned} d_{A\cup B}(T, \text{affordability}) &= (1-0.7)(0.30 \oplus 0.45) + 0.7(0.30 \cdot 0.45) \\ &= 0.3 \cdot 0.615 + 0.7 \cdot 0.135 \\ &= 0.279. \end{aligned}$$

Optional scalarization. With weights

$$w(\text{efficacy}) = 0.5, \qquad w(\text{safety}) = 0.3,$$

$$w(\text{affordability}) = 0.2,$$

$$\mu^{\star}_{A\cup B}(T) = 1 - \prod_{a \in P_v} (1 - w(a)\, d_{A\cup B}(T, a))$$

$$= 1 - (1 - 0.475)(1 - 0.22365)(1 - 0.0558) \approx 0.615.$$

The overall adoption score thus fuses heterogeneous evidence while respecting value–level contradiction.

2.3 DSm Vector Space of Refined Labels

A DSm Vector Space is a linear space over refined DSm labels; addition and scalar multiplication obey the classical axioms [16, 17].

Definition 2.5 (DSm Vector Space of Refined Labels) *Fix an integer $m \geq 1$ and let*

$$LR := \{L_r \mid r \in \mathbb{R}\}$$

be the set of refined labels*, equipped with the DSm field operations making $(LR, +, \times)$ isomorphic to $(\mathbb{R}, +, \cdot)$ via*

$$f_R : LR \longrightarrow \mathbb{R}, \qquad f_R(L_r) = \frac{r}{m+1}.$$

In particular, for $L_a, L_b \in LR$ one has

$$f_R(L_a + L_b) = f_R(L_a) + f_R(L_b)$$

and

$$f_R(L_a \times L_b) = f_R(L_a)\, f_R(L_b).$$

Scalar multiplication by real numbers is defined on labels by

$$\alpha \cdot L_a := L_{\alpha a}, \qquad (\alpha \in \mathbb{R}),$$

and extended componentwise to tuples, matrices, etc. of labels.

A DSm vector space of refined labels over $\mathbb{R}$ *is a set V of objects built from LR (e.g., LR^n, label matrices) together with*

(i) an addition

$$+ : V \times V \longrightarrow V$$

making $(V, +)$ an abelian group (typically componentwise label addition), and

(ii) an external scalar multiplication

$$\cdot : \mathbb{R} \times V \longrightarrow V$$

(given componentwise by $\alpha \cdot L_a = L_{\alpha a}$),

such that for all $\alpha, \beta \in \mathbb{R}$ and $u, v \in V$ the usual vector–space axioms hold:

$$\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v, \qquad (\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u,$$

$$(\alpha\beta) \cdot u = \alpha \cdot (\beta \cdot u), \qquad 1 \cdot u = u.$$

When, in addition, an internal (label) multiplication is defined on V that is distributive over $+$ (e.g., matrix product of label matrices), V becomes a DSm linear algebra of refined labels over $\mathbb{R}$.

Example 2.4 ***Workforce planning as a DSm vector space of refined labels*** *Fix $m = 9$ so that*

$$f_R(L_r) = \frac{r}{10}.$$

Consider three roles

$$\{D, I, Q\}$$

(Design, Implementation, QA) and the DSm vector space

$$V = LR^3$$

with componentwise addition and real scalar multiplication.

Effort vectors. *Let a phase–1 plan be*

$$u = (L_{12}, L_{25}, L_8) \quad \Longleftrightarrow \quad f_R(u) = (1.2,\ 2.5,\ 0.8)$$

(FTE–months).
A phase–2 adjustment is

$$v = (L_{-2}, L_{10}, L_4) \quad \Longleftrightarrow \quad f_R(v) = (-0.2,\ 1.0,\ 0.4).$$

Combining phases (vector addition). *Using label addition*

$$L_a + L_b = L_{a+b},$$

$$u + v = (L_{12-2}, L_{25+10}, L_{8+4}) = (L_{10}, L_{35}, L_{12}),$$

hence

$$f_R(u + v) = (1.0,\ 3.5,\ 1.2).$$

Interpretation: aggregated required efforts across the two phases.

Scaling for overtime (scalar multiplication). *With a* 50% *overtime plan,*

$$\alpha = 1.5\ :$$

$$\alpha u = (L_{1.5\cdot 12}, L_{1.5\cdot 25}, L_{1.5\cdot 8}) = (L_{18}, L_{37.5}, L_{12}),$$

so

$$f_R(\alpha u) = (1.8,\ 3.75,\ 1.2).$$

All vector axioms hold because f_R is an isomorphism; V is thus a DSm vector space modelling resource allocation with refined labels.

Example 2.5 ***Multi-sensor fusion in a smart city as a DSm vector space***

Fix $m = 9$ so

$$f_R(L_r) = \frac{r}{10}.$$

Monitor four normalized indicators (temperature anomaly, $PM_{2.5}$ index, noise level, congestion) and set

$$V = LR^4$$

with componentwise operations.

Two sensor passes.

First pass (satellite/ground blend):

$$x = (L_{32}, L_{15}, L_{40}, L_{22}) \quad \Longleftrightarrow \quad f_R(x) = (3.2,\ 1.5,\ 4.0,\ 2.2).$$

Second pass:

$$y = (L_{28}, L_{10}, L_{35}, L_{30}) \quad \Longleftrightarrow \quad f_R(y) = (2.8,\ 1.0,\ 3.5,\ 3.0).$$

Linear fusion (convex combination).
With confidence $\alpha = 0.3$ for x and 0.7 for y,

$$z = \alpha x + (1-\alpha) y = (L_{0.3\cdot 32+0.7\cdot 28}, L_{0.3\cdot 15+0.7\cdot 10}, L_{0.3\cdot 40+0.7\cdot 35}, L_{0.3\cdot 22+0.7\cdot 30}).$$

Compute the label parameters:

$$z = (L_{29.2}, L_{11.5}, L_{36.5}, L_{27.6}) \quad \Longleftrightarrow$$

$$f_R(z) = (2.92,\ 1.15,\ 3.65,\ 2.76).$$

This is the fused reading, obtained by vector operations in V.

Calibration (global gain).
If an instrument bias requires 10% *gain, multiply by $\beta = 1.1$:*

$$\beta z = (L_{1.1\cdot 29.2}, L_{1.1\cdot 11.5}, L_{1.1\cdot 36.5}, L_{1.1\cdot 27.6}) = (L_{32.12}, L_{12.65}, L_{40.15}, L_{30.36}),$$

so

$$f_R(\beta z) = (3.212,\ 1.265,\ 4.015,\ 3.036).$$

Because LR is (by definition) field-isomorphic to $\mathbb{R}$ through f_R, $V = LR^4$ satisfies all vector axioms and forms a DSm vector space suitable for linear sensor fusion and calibration in operational settings.

3. RESULTS AND DISCUSSIONS

This section presents the results of this paper.

3.1 DSm Fuzzy Vector Space

A DSm Fuzzy Vector Space assigns memberships in $[0, 1]$; sums combine via a t-norm, scalings attenuate consistently with fuzzy linearity.

Definition 3.1 (DSm Fuzzy Vector Space) *Let $(V, +, \cdot)$ be a DSm vector space over $\mathbb{R}$. A* DSm fuzzy vector space *is a pair (V, μ) where*

$$\mu\ :\ V \longrightarrow [0, 1]$$

satisfies, for all $u, v \in V$ and $\alpha \in \mathbb{R}$,

(F1) $\mu(0) = 1,$

(F2) $\mu(u + v) = \mu(u) \otimes \mu(v),$

(F3) $\mu(\alpha u) = \Lambda_{|\alpha|}(\mu(u)) = \mu(u)^{|\alpha|}.$

Example 3.1 ***Workforce allocation as a DSm fuzzy vector space of refined labels***
Fix $m = 9$ so that

$$f_R(L_r) = \frac{r}{10}.$$

Let

$$V = LR^3$$

model effort intensities for Design *(D),* Implementation *(I), and QA (Q). We take the product t-norm and the power family*

$$\Lambda_{|\alpha|}(x) = x^{|\alpha|},$$

and define

$$\mu(u) := \exp(-\|f_R(u)\|) \in [0,1], \qquad u \in V.$$

Because project effort coordinates are nonnegative by semantics, $\|\cdot\|_1$ is additive *(i.e. $\|u+v\|_1 = \|u\|_1 + \|v\|_1$) on the application domain, and the DSm fuzzy axioms hold as equalities:*

(F1) $\mu(0) = \exp(-0) = 1.$

(F2)

For

$$u = (L_{12}, L_{25}, L_8)$$

and

$$v = (L_3, L_2, L_5),$$

$$f_R(u) = (1.2, 2.5, 0.8), \qquad \|f_R(u)\|_1 = 4.5,$$
$$f_R(v) = (0.3, 0.2, 0.5), \qquad \|f_R(v)\|_1 = 1.0.$$

Then

$$u + v = (L_{15}, L_{27}, L_{13})$$

with

$$\|f_R(u+v)\|_1 = 5.5,$$

hence

$$\mu(u+v) = e^{-5.5} = e^{-4.5} \cdot e^{-1.0} = \mu(u) \otimes \mu(v).$$

(F3)

For

$$\alpha = 2,$$

$$\mu(\alpha u) = \exp(-\|f_R(2u)\|_1) = \exp(-2\|f_R(u)\|_1)$$
$$= (\exp(-\|f_R(u)\|_1))^2 = \mu(u)^{|\alpha|}.$$

Thus (V, μ) is a DSm fuzzy vector space modeling workforce allocation, where $\mu(u)$ encodes "confidence" (higher for smaller total effort).

Example 3.2 ***Cloud capacity planning as a DSm fuzzy vector space***
Fix $m = 9$ and set

$$V = LR^2$$

with coordinates Compute *(C) and* Storage *(S).*
Let

$$\lambda = 0.4,$$

choose the product t-norm and

$$\Lambda_{|\alpha|}(x) = x^{|\alpha|},$$

and define

$$\mu(u) := \exp(-\lambda\|f_R(u)\|_1), \qquad u \in V.$$

Because planned resource requests are nonnegative, $\|\cdot\|_1$ is additive on the domain, and

(F1) $\mu(0) = e^{-\lambda \cdot 0} = 1.$

(F2)

For

$$u = (L_{20}, L_{10})$$

and

$$v = (L_{15}, L_5),$$

$$f_R(u) = (2.0, 1.0), \qquad \|f_R(u)\|_1 = 3,$$

$$f_R(v) = (1.5, 0.5), \qquad \|f_R(v)\|_1 = 2.$$

Then

$$u + v = (L_{35}, L_{15})$$

with

$$\|f_R(u+v)\|_1 = 5,$$

hence

$$\mu(u+v) = e^{-4\cdot 0.5} = e^{-2.0} = \left(e^{-\lambda\cdot 3}\right)\left(e^{-\lambda\cdot 2}\right) = \mu(u)\otimes\mu(v).$$

(F3)

For

$$\alpha = \frac{3}{2},$$

$$\mu(\alpha u) = \exp\left(-\lambda \left\| f_R\left(\frac{3}{2}u\right)\right\|_1\right)$$

$$= \exp\left(-\lambda \cdot \frac{3}{2}\|f_R(u)\|_1\right) = \mu(u)^{|\alpha|}.$$

Therefore (V, μ) is a DSm fuzzy vector space quantifying the "acceptability" of a capacity plan (higher when the total requested load is smaller).

Theorem 3.1 ***DSm Fuzzy Vector Spaces generalize DSm Vector Spaces***
Every DSm vector space $(V, +, \cdot)$ embeds into a DSm fuzzy vector space by the injective map

$$i : V \longrightarrow \{(V,\mu) \text{ DSm fuzzy}\}, \qquad \mu_i(u) = 1 \quad (\forall u \in V).$$

Proof. Let $\mu \equiv 1$. Then (F1) holds trivially. Since 1 is the neutral element of any t-norm,

$$\mu(u+v) = 1 = 1\otimes 1 = \mu(u)\otimes\mu(v),$$

so (F2) is satisfied. For (F3),

$$\mu(\alpha u) = 1 = 1^{|\alpha|} = \Lambda_{|\alpha|}(1).$$

Thus (V, μ) is a DSm fuzzy vector space. Injectivity is clear from the definition of μ (the underlying V is unchanged). □

3.2 DSm Neutrosophic Vector Space

A DSm Neutrosophic Vector Space assigns triplets (T, I, F) to vectors; operations propagate truth, indeterminacy, and falsity via t-norms and aggregators appropriately.

Definition 3.2 (DSm Neutrosophic Vector Space) *Let $(V, +, \cdot)$ be as above. A* DSm neutrosophic vector space *is a quadruple*

$$(V, T, I, F)$$

with

$$T, I, F : V \longrightarrow [0, 1]$$

such that, for all $u, v \in V$ and $\alpha \in \mathbb{R}$,

(N1) $T(0) = 1,\ I(0) = 0,\ F(0) = 0,$

(N2) $T(u+v) = T(u)\otimes T(v), \qquad F(u+v) = F(u)\otimes F(v),$

(N3) $I(u+v) = I(u)\oplus I(v)$ *(e.g. take $\oplus = \max$),*

(N4) $T(\alpha u) = \Lambda_{|\alpha|}(T(u)), \quad I(\alpha u) = \Lambda_{|\alpha|}(I(u)),$ $F(\alpha u) = \Lambda_{|\alpha|}(F(u)).$

The triplet $(T(u), I(u), F(u))$ expresses degrees of truth/membership, indeterminacy, and falsity/nonmembership of u.

Example 3.3 ***Project workload planning as a DSm neutro sophic vector space*** *Fix $m = 9$ so that*

$$f_R(L_r) = \frac{r}{10}.$$

Let

$$V = LR^3$$

represent planned workload (in normalized units) for Design *(D),* Implementation *(I), and QA (Q). Addition and scalar multiplication are performed componentwise on labels, so $(V, +, \cdot)$ is a DSm vector space over $\mathbb{R}$.*

Neutrosophic structure. *Choose the product t-norm for T, the probabilistic sum*

$$a \oplus b = a + b - ab$$

for F and I, and set the scalar families

$$\Lambda^{(T)}_{|\alpha|}(x) = x^{|\alpha|}, \qquad \Lambda^{(F)}_{|\alpha|}(x) = \Lambda^{(I)}_{|\alpha|}(x) = 1 - (1-x)^{|\alpha|}.$$

For parameters $\lambda, \beta, \gamma > 0$, define for $u \in V$ (with nonnegative workloads so that $\|f_R(u)\|_1$ is additive)

$$T(u) = e^{-\lambda\|f_R(u)\|_1}, \qquad F(u) = 1 - e^{-\beta\|f_R(u)\|_1}, \qquad I(u) = 1 - e^{-\gamma\|f_R(u)\|_1}.$$

Axioms. *For all $u, v \in V$ and $\alpha \in \mathbb{R}$,*

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(N1) $T(0) = e^{-\lambda \cdot 0} = 1, \qquad I(0) = F(0) = 0.$

(N2)

$$\|f_R(u+v)\|_1 = \|f_R(u)\|_1 + \|f_R(v)\|_1,$$

$$T(u+v) = e^{-\lambda(\cdot)} = e^{-\lambda\|f_R(u)\|_1} e^{-\lambda\|f_R(v)\|_1} = T(u)\,T(v),$$

$$F(u+v) = 1 - e^{-\beta(\cdot)} = F(u) \oplus F(v),$$

$$I(u+v) = 1 - e^{-\gamma(\cdot)} = I(u) \oplus I(v).$$

(covered above with $\oplus$ for I).

(N4)

$$T(\alpha u) = e^{-\lambda|\alpha|\|f_R(u)\|_1} = \left(e^{-\lambda\|f_R(u)\|_1}\right)^{|\alpha|} = \Lambda^{(T)}_{|\alpha|}(T(u)),$$

$$F(\alpha u) = 1 - e^{-\beta|\alpha|\|f_R(u)\|_1} = 1 - (1 - F(u))^{|\alpha|} = \Lambda^{(F)}_{|\alpha|}(F(u)),$$

$$I(\alpha u) = \Lambda^{(I)}_{|\alpha|}(I(u)).$$

Numerical illustration. Let

$$\lambda = \beta = \gamma = 0.5,$$

$$u = (L_{10}, L_0, L_{20})$$

and

$$v = (L_5, L_{15}, L_0).$$

Then

$$f_R(u) = (1, 0, 2), \qquad \|f_R(u)\|_1 = 3;$$

$$f_R(v) = (0.5, 1.5, 0), \qquad \|f_R(v)\|_1 = 2;$$

$$\|f_R(u+v)\|_1 = 5.$$

Hence

$$T(u) = e^{-1.5} \approx 0.2231, \qquad T(v) = e^{-1} \approx 0.3679,$$

$$T(u+v) = e^{-2.5} \approx 0.0821 = T(u)T(v),$$

$$F(u) = 1 - e^{-1.5} \approx 0.7769, \qquad F(v) = 1 - e^{-1} \approx 0.6321,$$

$$F(u+v) = 1 - e^{-2.5} \approx 0.9179 = F(u) \oplus F(v),$$

and similarly for I.

Therefore (V, T, I, F) is a DSm neutrosophic vector space modeling project workloads: large planned effort lowers T (deliverability), increases F (delay risk) and I (uncertainty).

Example 3.4 ***Inventory replenishment planning as a DSm neutrosophic vector space*** *Let*

$$V = LR^3$$

encode daily replenishment requests for Batteries *(B),* Screens *(S), and* Chips *(C). As before,*

$$f_R(L_r) = \frac{r}{10},$$

and operations on V are componentwise, making $(V, +, \cdot)$ a DSm vector space.

Neutrosophic assignment. *Use the same aggregators as in the previous example (product t-norm for T, probabilistic sum for F and I) and the same scalar families*

$$\Lambda^{(T)}_{|\alpha|}(x) = x^{|\alpha|}, \qquad \Lambda^{(F)}_{|\alpha|} = \Lambda^{(I)}_{|\alpha|} = 1 - (1-x)^{|\alpha|},$$

but with parameters

$$\lambda = 0.3, \qquad \beta = 0.6, \qquad \gamma = 0.4\ :$$

$$T(u) = e^{-\lambda\|f_R(u)\|_1}, \qquad F(u) = 1 - e^{-\beta\|f_R(u)\|_1},$$

$$I(u) = 1 - e^{-\gamma\|f_R(u)\|_1}.$$

Interpretation. *$T(u)$ quantifies* on-time fulfillability *(high when requests are small), $F(u)$ the* stockout risk *(grows with total request), and $I(u)$ the uncertainty due to supplier variability.*

Check of axioms (symbolic). *For $u, v \in V$ with nonnegative coordinates,*

$$\|f_R(u+v)\|_1 = \|f_R(u)\|_1 + \|f_R(v)\|_1,$$

hence

$$T(u+v) = T(u)T(v), \qquad F(u+v) = F(u) \oplus F(v),$$

$$I(u+v) = I(u) \oplus I(v).$$

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and for any $\alpha \in \mathbb{R}$,

$$T(\alpha u) = \Lambda^{(T)}_{|\alpha|}(T(u)), \qquad F(\alpha u) = \Lambda^{(F)}_{|\alpha|}(F(u)),$$

$$I(\alpha u) = \Lambda^{(I)}_{|\alpha|}(I(u)).$$

Concrete values. Let

$$u = (L_{12}, L_8, L_{10})$$

and

$$v = (L_6, L_7, L_9).$$

Then

$$\|f_R(u)\|_1 = \frac{12+8+10}{10} = 3.0, \qquad \|f_R(v)\|_1 = \frac{6+7+9}{10} = 2.2,$$

$$\|f_R(u+v)\|_1 = 5.2.$$

Thus

$$T(u) = e^{-0.9} \approx 0.4066, \qquad T(v) = e^{-0.66} \approx 0.5150,$$

$$T(u+v) = e^{-1.56} \approx 0.2107 = T(u)T(v),$$

$$F(u) = 1 - e^{-1.8} \approx 0.8347, \qquad F(v) = 1 - e^{-1.32} \approx 0.7324,$$

$$F(u+v) = 1 - e^{-3.12} \approx 0.9562 = F(u) \oplus F(v),$$

$$I(u) = 1 - e^{-1.2} \approx 0.6988, \qquad I(v) = 1 - e^{-0.88} \approx 0.5866,$$

$$I(u+v) = 1 - e^{-2.08} \approx 0.8751 = I(u) \oplus I(v).$$

Therefore (V, T, I, F) satisfies (N1)–(N4) and provides a practical neutrosophic valuation of replenishment plans (truth for fulfillability, falsity for shortage risk, and indeterminacy for supply uncertainty).

Theorem 3.2 ***DSm Neutrosophic Vector Spaces generalize DSm Fuzzy Vector Spaces***

There is an injective, operation-preserving embedding

$$\mathcal{G}_{NF} : \{(V,\mu) \text{ DSm fuzzy}\} \hookrightarrow \{(V,T,I,F) \text{ DSm neutrosophic}\}$$

given by

$$T(u) := \mu(u), \qquad I(u) := 0, \qquad F(u) := 1 - \mu(u).$$

Proof. Let (V,μ) be DSm fuzzy. Then (N1) follows from $\mu(0) = 1$:

$$T(0) = 1, \qquad I(0) = 0, \qquad F(0) = 1 - \mu(0) = 0.$$

For addition, using (F2) and the fact that $\oplus$ is the De Morgan dual of $\otimes$,

$$T(u+v) = \mu(u+v) = \mu(u) \otimes \mu(v) = T(u) \otimes T(v),$$

$$F(u+v) = 1 - \mu(u+v) = 1 - (\mu(u) \otimes \mu(v))$$

$$= (1 - \mu(u)) \oplus (1 - \mu(v)) = F(u) \oplus F(v).$$

For indeterminacy,

$$I(u+v) = 0 = 0 \oplus 0 = I(u) \oplus I(v),$$

so (N3) holds (e.g. with $\oplus = \max$). For scalars, (F3) gives

$$T(\alpha u) = \mu(\alpha u) = \Lambda_{|\alpha|}(\mu(u)) = \Lambda_{|\alpha|}(T(u)),$$

$$I(\alpha u) = 0 = \Lambda_{|\alpha|}(0),$$

$$F(\alpha u) = 1 - \mu(\alpha u) = 1 - \Lambda_{|\alpha|}(\mu(u)).$$

Since

$$1 - r = \Lambda_{|\alpha|}(1 - r)$$

if we define $F(\alpha u)$ via $\Lambda_{|\alpha|}$ as well, we declare (as in (N4))

$$F(\alpha u) = \Lambda_{|\alpha|}(F(u)),$$

which equals

$$1 - \Lambda_{|\alpha|}(\mu(u))$$

because $\Lambda_{|\alpha|}$ is order-preserving on $[0,1]$ and distributes over standard negation when taken as $r^{|\alpha|}$. Hence (N4) holds. Injectivity is clear because $T = \mu$ recovers μ. □

3.3 DSm Plithogenic Vector Space

A DSm plithogenic vector space uses attribute-valued degrees and contradiction; mixer

$$(1-c)\otimes + c\oplus$$

unifies fuzzy and neutrosophic behaviors simultaneously.

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Definition 3.3 ***DSm Plithogenic Vector Space***

Fix a nonempty attribute value set P_v *with a designated dominant value*

$$v^\star \in P_v,$$

and a contradiction map

$$c : P_v \longrightarrow [0,1]$$

(interpreted as the degree of contradiction of a value w.r.t. $v^\star$).

A DSm plithogenic vector space *is a triple*

$$(V, d, c)$$

where

$$d : V \times P_v \longrightarrow [0,1]$$

satisfies, for all $u, v \in V$, $\alpha \in \mathbb{R}$, and $a \in P_v$,

$$\begin{aligned} &(P1) \quad d(0,a) = 1, \\ &(P2) \quad d(u+v,a) = (1-c(a))\big(d(u,a) \otimes d(v,a)\big) \\ &\qquad\quad +c(a)\big(d(u,a) \oplus d(v,a)\big), \\ &(P3) \quad d(\alpha u, a) = \Lambda_{|\alpha|}\big(d(u,a)\big). \end{aligned}$$

The map $d(\cdot, a)$ is the degree of appurtenance of vectors to the value a, and $c(a)$ mixes conjunctive ($\otimes$) and disjunctive ($\oplus$) behaviors according to plithogenic contradiction.

Example 3.5 ***Portfolio allocation as a DSm plithogenic vector space***

Ambient DSm space. *Fix $m = 9$ and refined labels*

$$LR = \{L_r : r \in \mathbb{R}\}$$

with

$$f_R(L_r) = \frac{r}{10}.$$

Let

$$V := LR^3$$

encode a portfolio allocation across Stocks (S), Bonds (B), and Cash (C). Vector addition and real scalar multiplication act componentwise on labels, so $(V, +, \cdot)$ is a DSm vector space over $\mathbb{R}$.

Plithogenic ingredients. *Let the attribute value set*

$$P_v = \{Return, Risk, Liquidity\}$$

with dominant value

$$v^\star = Return$$

and a contradiction map (qualitative priorities)

$$c(Return) = 0, \qquad c(Risk) = 0.9, \qquad c(Liquidity) = 0.4.$$

Choose a single binary combiner for both t-norm and s-norm:

$$u \otimes v = u \oplus v = uv, \qquad (u, v \in [0,1]),$$

and the scalar family

$$\Lambda_{|\alpha|}(x) = x^{|\alpha|}.$$

For each attribute

$$a \in P_v,$$

fix a sensitivity

$$k_a > 0$$

and define, for

$$u = (u_S, u_B, u_C) \in V,$$

$$\|u\|_1 := \|f_R(u)\|_1 = \sum_{j \in \{S,B,C\}} |f_R(u_j)|,$$

$$d(u,a) := \exp(-k_a \|u\|_1).$$

Then

$$d(0,a) = 1$$

(P1),

$$d(\alpha u, a) = (\exp(-k_a\|u\|_1))^{|\alpha|} = \Lambda_{|\alpha|}(d(u,a))$$

(P3), and

$$d(u+v,a) = \exp(-k_a(\|u\|_1 + \|v\|_1))$$

$$= \exp(-k_a\|u\|_1)\,\exp(-k_a\|v\|_1) = d(u,a) \otimes d(v,a),$$

so (P2) holds because the plithogenic mixer reduces to

$$(1-c) \otimes +c\oplus = \otimes$$

when $\otimes = \oplus$.

Numerics. *Let*

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$$u = (L_{15}, L_{10}, L_5), \qquad v = (L_5, L_0, L_{10}).$$

Then

$$\|u\|_1 = \frac{15+10+5}{10} = 3.0, \qquad \|v\|_1 = \frac{5+0+10}{10} = 1.5,$$
$$\|u+v\|_1 = 4.5.$$

Choose

$$(k_{Return}, k_{Risk}, k_{Liquidity}) = (0.4, 0.8, 0.5).$$

Then

$$d(u, \mathrm{Return}) = e^{-1.2} \approx 0.30119,$$
$$d(v, \mathrm{Return}) = e^{-0.6} \approx 0.54881,$$

$$d(u+v, \mathrm{Return}) = e^{-1.8} \approx 0.16530 = 0.30119 \times 0.54881$$
$$= d(u, \mathrm{Return}) \otimes d(v, \mathrm{Return}),$$

$$d(u, \mathrm{Risk}) = e^{-2.4} \approx 0.09072, \qquad d(v, \mathrm{Risk}) = e^{-1.2} \approx 0.30119,$$

$$d(u+v, \mathrm{Risk}) = e^{-3.6} \approx 0.02732 = 0.09072 \times 0.30119$$
$$= d(u, \mathrm{Risk}) \otimes d(v, \mathrm{Risk}).$$

For

$$\alpha = 2,$$

$$d(2u, \mathrm{Return}) = e^{-2.4} \approx 0.09072 = \left(e^{-1.2}\right)^2 = \Lambda_2(d(u, \mathrm{Return})).$$

Thus (V, d, c) is a valid DSm plithogenic vector space modeling portfolio membership degrees per attribute; the contradiction map c is encoded structurally and would influence the mixer if distinct $(\otimes, \oplus)$ were used.

Example 3.6 ***Renewable dispatch planning as a DSm plithogenic vector space***
Ambient DSm space. Fix $m = 9$ and work in

$$V := LR^2,$$

where

$$u = (u_W, u_S)$$

represents planned Wind *and* Solar *outputs (normalized). As before,*

$$f_R(L_r) = \frac{r}{10}$$

and $(V, +, \cdot)$ is a DSm vector space.

Plithogenic ingredients. *Let*

$$P_v = \{Stability,\ CostSaving,\ Variability\}$$

with dominant

$$v^\star = Stability,$$

and contradiction

$$c(Stability) = 0, \qquad c(CostSaving) = 0.4,$$
$$c(Variability) = 0.9.$$

Take again

$$\otimes = \oplus = product$$

and

$$\Lambda_{|\alpha|}(x) = x^{|\alpha|}.$$

For attribute-wise sensitivities

$$(k_{Stab}, k_{Cost}, k_{Var}) = (0.5, 0.3, 1.0),$$

define, for $u \in V$,

$$\|u\|_1 = \|f_R(u)\|_1, \qquad d(u, a) = \exp(-k_a \|u\|_1).$$

Then (P1)–(P3) hold for the same reasons as in Example 3.10.

Numerics. Let

$$u = (L_{12}, L_8)$$

and

$$v = (L_6, L_4),$$

so

$$\|u\|_1 = \frac{12+8}{10} = 2.0, \qquad \|v\|_1 = \frac{6+4}{10} = 1.0, \qquad \|u+v\|_1 = 3.0.$$

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a	$d(u,a)$	$d(v,a)$	$d(u+v,a)$ vs. $d(u,a)\otimes d(v,a)$
Stability	$e^{-0.5\cdot 2}=e^{-1}\approx 0.36788$	$e^{-0.5\cdot 1}=e^{-0.5}\approx 0.60653$	$e^{-1.5}\approx 0.22313=0.36788\times 0.60653$
CostSaving	$e^{-0.3\cdot 2}=e^{-0.6}\approx 0.54881$	$e^{-0.3\cdot 1}=e^{-0.3}\approx 0.74082$	$e^{-0.9}\approx 0.40657=0.54881\times 0.74082$
Variability	$e^{-1.0\cdot 2}=e^{-2}\approx 0.13534$	$e^{-1.0\cdot 1}=e^{-1}\approx 0.36788$	$e^{-3}\approx 0.04979=0.13534\times 0.36788$

For scaling, with

$$\alpha = 2$$

and

$$a = \text{Stability},$$

$$d(2u,\text{Stability}) = e^{-0.5\cdot 4} = e^{-2} \approx 0.13534$$

$$= \left(e^{-1}\right)^2 = \Lambda_2(d(u,\text{Stability})).$$

Hence $(V; d, c)$ provides a DSm plithogenic vector-space model of renewable dispatch, where each attribute value quantifies a different operational concern; the chosen $(\otimes,\oplus)$ makes the contradiction weighting inert, but the structure remains ready to capture it when nonidentical norms are adopted.

Theorem 3.3 ***DSm Plithogenic Vector Spaces generalize both DSm Fuzzy and DSm Neutrosophic Vector Spaces***
There are injective embeddings

$$\mathcal{G}_{PF}\ :\ \{(V,\mu)\ \textit{DSm fuzzy}\} \hookrightarrow \{(V;d,c)\ \textit{DSm plithogenic}\},$$

$$\mathcal{G}_{PN}\ :\ \{(V;T,I,F)\ \textit{DSm neutrosophic}\}$$

$$\hookrightarrow \{(V;d,c)\ \textit{DSm plithogenic}\},$$

defined as follows:

(F)

(Fuzzy case) take

$$P_v = \{*\},$$

$$c(*) = 0,$$

and set

$$d(u,*) := \mu(u).$$

Then, for all u, v and α,

$$d(u+v,*) = (1-0)(\mu(u)\otimes\mu(v)) + 0(\mu(u)\oplus\mu(v)) = \mu(u)\otimes\mu(v),$$

$$d(\alpha u,*) = \Lambda_{|\alpha|}(\mu(u)),$$

which reproduces (F2)–(F3). Thus $\mathcal{G}_{PF}$ is well-defined and injective.

(N)

(Neutrosophic case) take

$$P_v = \{T, I, F\},$$

choose the contradiction levels

$$c(T) = 0, \qquad c(F) = 1, \qquad c(I) = 1,$$

and define

$$d(u,T) := T(u), \qquad d(u,I) := I(u), \qquad d(u,F) := F(u).$$

Then

$$\begin{aligned} d(u+v,T) &= (1-0)(T(u)\otimes T(v)) + 0(T(u)\oplus T(v)) \\ &= T(u)\otimes T(v), \end{aligned}$$

$$\begin{aligned} d(u+v,F) &= (1-1)(F(u)\otimes F(v)) + 1(F(u)\oplus F(v)) \\ &= F(u)\oplus F(v), \end{aligned}$$

$$\begin{aligned} d(u+v,I) &= (1-1)(I(u)\otimes I(v)) + 1(I(u)\oplus I(v)) \\ &= I(u)\oplus I(v), \end{aligned}$$

and

$$d(\alpha u, a) = \Lambda_{|\alpha|}(d(u,a)), \qquad (a \in \{T, I, F\}),$$

which coincides with (N2)–(N4). Hence $\mathcal{G}_{PN}$ is an injective embedding.

Proof. Both parts are direct computations from the definitions of d, c, and the plithogenic mixer in (P2). Injectivity in each case is immediate since the original membership(s) are recovered componentwise from d. □

4. CONCLUSIONS

In this paper, we newly introduced the extensions of DSm Vector Space using Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets, namely the DSm Fuzzy Vector Space, the DSm Neutrosophic Vector Space, and the DSm Plithogenic Vector Space, and investigate their structural properties. Looking ahead, we expect further investigations into extended frameworks built upon HyperGraph [36], Rough Set [37, 38], Soft Set [39, 40], SuperHyperGraph [32, 41], and HyperVector Space [42, 43] among related approaches. We also expect future research to advance through computational experiments and related empirical investigations.

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