



Research Paper

Upper and Lower Approximations of Subsemimodules in Semimodules Over Semirings

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Abstract

This paper develops a rough set-theoretic framework for semimodules over commutative semirings using the Bourne relation induced by subsemimodules and partitioning subsemimodules. Upper and lower approximations of subsets and subsemimodules are introduced and systematically analyzed. Necessary and sufficient conditions under which these approximations constitute subsemimodules are established, with particular emphasis on subtractive extensions. Several examples are provided to illustrate the theory. The results extend rough set theory to semimodule contexts and furnish a coherent algebraic framework for additive inverse-free structures.

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1. INTRODUCTION

The concept of rough set is introduced by Pawlak [1]. The concepts of lower and upper approximations of a set is studied by Pawlak [2]. A pair (U, θ) , where U is a non-empty set and θ is an equivalence relation on U , is called an approximation space. Denote $P(U)$ = the set of all subsets of U . For an approximation space (U, θ) , by a rough approximation in (U, θ) we mean a mapping $Apr : P(U) \rightarrow P(U) \times P(U)$ defined by for every $X \in P(U)$, $Apr(X) = (Apr(X), \overline{Apr}(X))$ where $Apr(X) = \{x \in U : [x]_\theta \subseteq X\}$ and $\overline{Apr}(X) = \{x \in U : [x]_\theta \cap X \neq \emptyset\}$. $Apr(X)$ is called a lower rough approximation of X in (U, θ) , where as $\overline{Apr}(X)$ is called upper rough approximation of X in (U, θ) . Banikowaski [3] has studied algebraic structures of rough sets. Some works on rough ideals in semigroups may be found in [4, 5]. Theory of roughness in rings and modules is studied by Davvaz [6, 7]. Theory of rough ideals of a semirings with respect to the Bourne relation induced by an ideal of a semiring is studied by Selvan and Senthil Kumar [8]. Rough prime ideals, rough primary ideals and fuzzy prime ideals in commutative rings are studied in Davvaz [9]. The concept of subtractive extension of an ideal in a semiring is recently introduced and studied by Chaudhari and Bonde [10].

Semirings and semimodules provide a natural algebraic setting for extending rough set theory, particularly in subtractive-inverse-free contexts where classical ring-based structures are inadequate. Many modern applications such as tropical algebra, weighted automata, and information systems are inherently semiring-driven, making this framework both structurally necessary and application motivated. The Bourne congruence relation, arising from subtractive subsemimodules, induces natural partitions of semimodules and enables quotient-like structures even without additive inverses. Complementarily, partitioning subsemimodules yields stratified layers that support granular analysis of approximation processes. Together, these mechanisms allow coherent definitions of upper and lower approximations, rough subsemimodules, and rough prime or primary structures. This approach significantly broadens the expressive power of rough set theory and deepens its algebraic foundations within semiring-based systems.

For the definitions of monoid and semiring, we refer to Golan [11]. All semirings in this paper are commutative with nonzero identity. $\mathbb{Z}_0^+(\mathbb{N})$ will denote the set of all non-negative (positive) integers. An ideal I of a semiring R is called a subtractive ideal (= k -ideal) if $a, a + b \in I, b \in R$, then $b \in I$. Let R be a semiring. A left R -semimodule is a commutative monoid $(M, +)$ with additive

identity 0_M equipped with a scalar multiplication $R \times M \rightarrow M$, $(r, x) \mapsto rx$, satisfying the following conditions for all $r, r' \in R$ and $x, y \in M$:

1. $(rr')x = r(r'x)$;
2. $r(x + y) = rx + ry$;
3. $(r + r')x = rx + r'x$;
4. $1_Rx = x$;
5. $r0_M = 0_M = 0_Rx$.

A nonempty subset N of a left R -semimodule M is called a *subsemimodule* if it is closed under addition and scalar multiplication. Throughout, the term R -semimodule refers to a left semimodule over a commutative semiring R . Every semiring R is a $(\mathbb{Z}_0^+, +, \cdot)$ -semimodule [11].

Definition 2.1 [[12], Definition 2.1] A subsemimodule N of M is called *partitioning* (or a Q -subsemimodule) if there exists a subset $Q \subseteq M$ such that:

- i) $M = \cup\{q + N : q \in Q\}$;
- ii) for $q_1, q_2 \in Q$, $(q_1 + N) \cap (q_2 + N) = \emptyset$ if and only if $q_1 = q_2$.

If N is a partitioning subsemimodule of an R -semimodule M , then $M/N_{(Q)} = \{q + N : q \in Q\}$ forms an R -semimodule under the addition operation \oplus , defined by $(q_1 + N) \oplus (q_2 + N) = q_3 + N$, where $q_3 \in Q$ is the unique element satisfying $q_1 + q_2 + N \subseteq q_3 + N$. Similarly, scalar multiplication \odot is defined by $r \odot (q_1 + N) = q_4 + N$, where $q_4 \in Q$ is the unique element such that $r q_1 + N \subseteq q_4 + N$. The R -semimodule $M/N_{(Q)}$ is called the *quotient semimodule* of M by N [4].

The following results will be used frequently in the sequel.

Lemma 1.1 ([12], [Lemma 1.3]) Let N be a subsemimodule of an R -semimodule M and $x, y \in M$ such that $x + N \subseteq y + N$. Then for all $z \in M$ and $r \in R$, $x + z + N \subseteq y + z + N$, and $rx + N \subseteq ry + N$.

Lemma 1.2 ([12], [Lemma 2.2]) Let N be a Q -subsemimodule of an R -semimodule M . For each $x \in M$, there exists a unique $q \in Q$ such that $x + N \subseteq q + N$. Hence $x = q + a$ for some $a \in N$.

Theorem 1.1 [12]

Let N be a Q -subsemimodule of an R -semimodule M and let A be a subtractive subsemimodule of M with $N \subseteq A$. Then:

- 1) N is a $(Q \cap A)$ -subsemimodule of A ;
- 2) $A/N_{(Q \cap A)} = \{q + N : q \in Q \cap A\}$ is a subtractive subsemimodule of $M/N_{(Q)}$.

A subsemimodule N of R -semimodule M is called *subtractive* (or k -subsemimodule) if $x, x + y \in N$ with $y \in M$ implies $y \in N$.

2. UPPER AND LOWER APPROXIMATIONS OF SUBSEMIMODULES WITH RESPECT TO BOURNE RELATION

Selvan and Senthil Kumar [8] introduced the notion of rough ideals of a semiring with respect to the Bourne relation induced by an ideal of a semiring. Equivalence classes induced by an equivalence relation form a partition for a given set, and a rough set is defined on this partition induced by an equivalence relation.

A subsemimodule N of an R -semimodule M defines an equivalence relation \equiv_N on M , called the Bourne relation, given by $m \equiv_N m'$ if and only if there exists $n, n' \in N$ such that $m + n = m' + n'$. Note that if $m \equiv_N m'$ and $s \equiv_N s'$ for some $m, m', s, s' \in M$ and $r \in R$, then $m + s \equiv_N m' + s'$ and $rm \equiv_N rm'$. The set of all equivalence classes of elements of M with respect to a subsemimodule N of an R -semimodule M under the Bourne relation is denoted by M/N and the equivalence class of an element m of M with respect to a subsemimodule N of an R -semimodule M is denoted by $[m]_N^M$. The upper and lower approximations of a subset X of an R -semimodule M with respect to an equivalence relation \equiv_N are denoted by

- 1) $\overline{Apr}_N(X) = \{x \in M : [x]_N^M \cap X \neq \emptyset\}$;
- 2) $\underline{Apr}_N(X) = \{x \in M : [x]_N^M \subseteq X\}$.

For a subset X of M , if $\underline{Apr}_N(X)$ and $\overline{Apr}_N(X)$ are subsemimodules, prime subsemimodules and primary subsemimodules of an R -semimodule M , then $(\underline{Apr}_N(X), \overline{Apr}_N(X))$ is called respectively a rough subsemimodule, rough prime subsemimodule and rough primary subsemimodule of an R -semimodule M .

Note that $\underline{Apr}_N(X) \subseteq X \subseteq \overline{Apr}_N(X)$.

Example 1. Consider $M = \mathbb{Z}_0^+$ is a semimodule over the semiring $R = (\mathbb{Z}_0^+, +, \cdot)$ and subsemimodule $N = \langle 4 \rangle = 4\mathbb{Z}_0^+$. Then

- 1) If $X = \langle 2 \rangle = 2\mathbb{Z}_0^+$, then $\underline{Apr}_N(X) = \{0, 4, 8, \dots\} \cup \{2, 6, 10, \dots\} = [0]_N^M \cup [2]_N^M = 2\mathbb{Z}_0^+$ and $\overline{Apr}_N(X) = \{0, 4, 8, \dots\} \cup \{2, 6, 10, \dots\} = [0]_N^M \cup [2]_N^M = 2\mathbb{Z}_0^+$;
- 2) If $X = \{1, 2, 3\}$, then $\underline{Apr}_N(X) = \emptyset$ and $\overline{Apr}_N(X) = \{1, 5, 9, \dots\} \cup \{2, 6, 10, \dots\} \cup \{3, 7, 11, \dots\} = [1]_N^M \cup [2]_N^M \cup [3]_N^M = \{1, 2, 3, 5, 6, 7, 9, \dots\}$. Here $\underline{Apr}_N(X)$ and $\overline{Apr}_N(X)$ are not subsemimodules;
- 3) If $X = \langle 3 \rangle = 3\mathbb{Z}_0^+$, then $\underline{Apr}_N(X) = \emptyset$ and $\overline{Apr}_N(X) = [0]_N^M \cup [1]_N^M \cup [2]_N^M = \mathbb{Z}_0^+$.

Definition 2. Let N be a subsemimodule of an R -semimodule M . A subsemimodule A of M with $N \subseteq A$ is said to be a *subtractive extension* of N if $x \in N, x + y \in A, y \in M$, then $y \in A$.

Clearly, every subtractive subsemimodule of an R -semimodule M containing a subsemimodule N of M is a subtractive extension of N .

Theorem 2.1 Let N, K be subsemimodules of an R semimodule M with $N \subseteq K$. Then K is a subtractive extension of N if and only if $\overline{Apr}_N(K) = K$.

Proof. First, suppose that K is a subtractive extension of N . Clearly, $K \subseteq \overline{Apr}_N(K)$. For the other inclusion, let $x \in \overline{Apr}_N(K)$. Then $[x]_N^M \cap K \neq \emptyset$. Let $a \in [x]_N^M \cap K$. As $a \in [x]_N^M, a + n = x + n'$ for some $n, n' \in N \subseteq K$. Now, $x + n' = a + n \in K, n' \in N$ and K is a subtractive extension of N , it follows that $x \in K$. Hence $\overline{Apr}_N(K) \subseteq K$. Conversely, suppose that $\overline{Apr}_N(K) = K$. Let $x \in N$ and $x + y \in K$ for some $y \in M$. Now $x + y \in K$ implies $x + y = k + 0$ where $k \in K$ and hence $y \in [k]_N^M$. Since $k \in [k]_N^M \cap K, [k]_N^M \cap K \neq \emptyset$. Now $y \in [k]_N^M \Rightarrow [y]_N^M \cap K \neq \emptyset$. Hence $y \in \overline{Apr}_N(K) = K$ as needed.

Corollary 2.1 Let N, K be subsemimodules of an R -semimodule M such that $N \subseteq K$. If K is a subtractive subsemimodule, then $\overline{Apr}_N(K) = K$.

The following example shows that the converse of the above corollary is not true.

Example 2. Consider $M = \mathbb{Z}_0^+$ is a semimodule over a semiring $R = (\mathbb{Z}_0^+, +, \cdot)$ and subsemimodules $N = \{0\}$ and $K = \langle 2, 3 \rangle = \mathbb{Z}_0^+ - \{1\}$ of M . Here $\overline{Apr}_N(K) = K$. But K is not a subtractive subsemimodule of M , as $2, 2 + 1 \in K$ and $1 \notin K$.

Theorem 2.2 If K is a subtractive extension of a subsemimodule N of an R -semimodule M , then $\overline{Apr}_N(K)$ is a subsemimodule of M .

Proof. Clearly, $\overline{Apr}_N(K) \neq \emptyset$. Let $a, b \in \overline{Apr}_N(K)$ and $r \in R$. Then $a \in [a]_N^M \subseteq K, b \in [b]_N^M \subseteq K$. Since K is a subsemimodule of $M, a + b \in K$ and $ra \in K$. Let $x \in [a + b]_N^M$. Therefore, $x + n_1 = a + b + n_2$ for some $n_1, n_2 \in N$. Since $a + b \in K, N \subseteq K$ and K is subtractive extension of $N, x \in K$. Hence $[a + b]_N^M \subseteq K$. Thus $a + b \in \overline{Apr}_N(K)$. Similarly $ra \in \overline{Apr}_N(K)$. Hence $\overline{Apr}_N(K)$ is a subsemimodule of M .

Theorem 2.3 Let N, K be subsemimodules of an R semimodule M with $N \subseteq K$. Then K is a subtractive extension of N if and only if $\overline{Apr}_N(K) = K$.

Proof. First, suppose that K is a subtractive extension of N . Clearly, $\overline{Apr}_N(K) \subseteq K$. For the other inclusion, let $x \in K$ and $a \in [x]_N^M$. Therefore, $a + n = x + n'$ for some $n, n' \in N \subseteq K$. Now, $a + n = x + n' \in K, n \in N$ and K is a subtractive extension of $N \Rightarrow a \in K$. Hence $[x]_N^M \subseteq K$. So $x \in \overline{Apr}_N(K)$. Hence $K \subseteq \overline{Apr}_N(K)$. Conversely, suppose that $\overline{Apr}_N(K) = K$. Let $x \in N$ and $x + y \in K$ for some $y \in M$. Now $x + y \in K$ implies $x + y = k + 0$ where $k \in K$ and hence $y \in [k]_N^M \subseteq K$ as needed.

Corollary 2.2 Let N, K be subsemimodules of an R semimodule M with $N \subseteq K$. If K is a subtractive subsemimodule, then $\overline{Apr}_N(K) = K$.

The following example shows that the converse of the above corollary is not true.

Example 3 Consider $M = \mathbb{Z}_0^+$ is a semimodule over a semiring $R = (\mathbb{Z}_0^+, +, \cdot)$ and subsemimodules $N = \{0\}$ and $K = \langle 2, 3 \rangle = \mathbb{Z}_0^+ - \{1\}$ of M . Here $\overline{Apr}_N(K) = K$. But K is not a subtractive subsemimodule of M .

3. ROUGH SETS USING PARTITIONING SUBSEMIMODULES

Let N be a submodule of an R -module M . For $x, y \in M$, we say that x is congruent to y (mod N), written as $x \equiv y$ (mod N) if $x - y \in N$. It is easy to see that the relation $x \equiv y$ (mod N) is an equivalence relation. Therefore, a submodule N of M induces an equivalence relation on M . But this is not true in semimodule over semiring for any subsemimodule. Chaudhari and Bonde

[12], defined a restricted class of subsemimodules in semimodules over semirings known as partitioning subsemimodule (= Q -subsemimodule) which forms a partition for a semimodule over semiring. Using this concept, we introduce the notion of rough subsemimodule induced by a partitioning subsemimodule of a semimodule over semiring. For a Q -subsemimodule N of an R -semimodule M , the upper and lower approximations of a subset X of R are given by,

- 1) $\overline{Apr}_{N(Q)}(X) = \{x \in M : x \in q + N \text{ for some } q \in Q \text{ and } (q + N) \cap X \neq \emptyset\};$
- 2) $\underline{Apr}_{N(Q)}(X) = \{x \in M : x \in q + N \text{ for some } q \in Q \text{ and } q + N \subseteq X\}.$

Theorem 3.1 Let N be a Q -subsemimodule of an R -semimodule M and A, B be subsets of M . Then:

- i) $\underline{Apr}_{N(Q)}(A) \subseteq A \subseteq \overline{Apr}_{N(Q)}(A);$
- ii) $\underline{Apr}_{N(Q)}(\emptyset) = \emptyset = \overline{Apr}_{N(Q)}(\emptyset);$
- iii) $\underline{Apr}_{N(Q)}(M) = M = \overline{Apr}_{N(Q)}(M);$
- iv) If $A \subseteq B$, then $\underline{Apr}_{N(Q)}(A) \subseteq \underline{Apr}_{N(Q)}(B)$ and $\overline{Apr}_{N(Q)}(A) \subseteq \overline{Apr}_{N(Q)}(B);$
- v) $\underline{Apr}_{N(Q)}(\underline{Apr}_{N(Q)}(A)) = \underline{Apr}_{N(Q)}(A);$
- vi) $\overline{Apr}_{N(Q)}(\overline{Apr}_{N(Q)}(A)) = \overline{Apr}_{N(Q)}(A);$
- vii) $\underline{Apr}_{N(Q)}(\overline{Apr}_{N(Q)}(A)) = \underline{Apr}_{N(Q)}(A);$
- viii) $\overline{Apr}_{N(Q)}(\underline{Apr}_{N(Q)}(A)) = \overline{Apr}_{N(Q)}(A);$
- ix) $\underline{Apr}_{N(Q)}(A) = (\overline{Apr}_{N(Q)}(A^c))^c;$
- x) $\overline{Apr}_{N(Q)}(A) = (\underline{Apr}_{N(Q)}(A^c))^c;$
- xi) $\underline{Apr}_{N(Q)}(A \cap B) = \underline{Apr}_{N(Q)}(A) \cap \underline{Apr}_{N(Q)}(B);$
- xii) $\overline{Apr}_{N(Q)}(A \cap B) \subseteq \overline{Apr}_{N(Q)}(A) \cap \overline{Apr}_{N(Q)}(B);$
- xiii) $\underline{Apr}_{N(Q)}(A \cup B) \supseteq \underline{Apr}_{N(Q)}(A) \cup \underline{Apr}_{N(Q)}(B);$
- xiv) $\overline{Apr}_{N(Q)}(A \cup B) = \overline{Apr}_{N(Q)}(A) \cup \overline{Apr}_{N(Q)}(B);$
- xv) $\underline{Apr}_{N(Q)}(q + N) = \overline{Apr}_{N(Q)}(q + N) = q + N$ for all $q \in Q$.

Proof. i) Let $x \in \underline{Apr}_{N(Q)}(A)$. Then there exists $q \in Q$ such that $x \in q + N$ and $q + N \subseteq A$. So $\underline{Apr}_{N(Q)}(A) \subseteq A$. Now, let $a \in A \subseteq M$. Then there exists $q \in Q$ such that $a \in q + N$. Now $a \in (q + N) \cap A \Rightarrow (q + N) \cap A \neq \emptyset \Rightarrow a \in \overline{Apr}_{N(Q)}(A)$.

ii) and iii) Trivial.

iv) $x \in \underline{Apr}_{N(Q)}(A) \Rightarrow x \in q + N$ for some $q \in Q$ and $q + N \subseteq A \subseteq B \Rightarrow x \in \underline{Apr}_{N(Q)}(B)$. Hence $\underline{Apr}_{N(Q)}(A) \subseteq \underline{Apr}_{N(Q)}(B)$. Also, $y \in \overline{Apr}_{N(Q)}(A) \Rightarrow y \in q + N$ for some $q \in Q$ and $(q + N) \cap A \neq \emptyset \Rightarrow y \in q + N$ and $(q + N) \cap B \neq \emptyset \Rightarrow y \in \overline{Apr}_{N(Q)}(B)$. Hence $\overline{Apr}_{N(Q)}(A) \subseteq \overline{Apr}_{N(Q)}(B)$.

v) From i), $\underline{Apr}_{N(Q)}(\underline{Apr}_{N(Q)}(A)) \subseteq \underline{Apr}_{N(Q)}(A)$. Assume that $x \in \underline{Apr}_{N(Q)}(A)$. Then $x \in q + N$ for some $q \in Q$ and $q + N \subseteq A$. But then $q + N \subseteq \underline{Apr}_{N(Q)}(A)$. So $x \in \underline{Apr}_{N(Q)}(\underline{Apr}_{N(Q)}(A))$. This implies $\underline{Apr}_{N(Q)}(A) \subseteq \underline{Apr}_{N(Q)}(\underline{Apr}_{N(Q)}(A))$. Thus, $\underline{Apr}_{N(Q)}(\underline{Apr}_{N(Q)}(A)) = \underline{Apr}_{N(Q)}(A)$.

- vi) From (i), $\overline{Apr}_{N(Q)}(A) \subseteq \overline{Apr}_{N(Q)}(\overline{Apr}_{N(Q)}(A))$. For the other inclusion, let $x \in \overline{Apr}_{N(Q)}(\overline{Apr}_{N(Q)}(A))$. Then $x \in q + N$ and $(q + N) \cap \overline{Apr}_{N(Q)}(A) \neq \emptyset$ for some $q \in Q$. Let $y \in (q + N) \cap \overline{Apr}_{N(Q)}(A)$. Then $y \in q' + N$ for some $q' \in Q$ and $(q' + N) \cap A \neq \emptyset$. Now $y \in (q + N) \cap (q' + N)$ implies $q = q'$. So $(q + N) \cap A \neq \emptyset$. Hence $x \in \overline{Apr}_{N(Q)}(A)$. This implies $\overline{Apr}_{N(Q)}(\overline{Apr}_{N(Q)}(A)) \subseteq \overline{Apr}_{N(Q)}(A)$.
- vii) From (i), $\underline{Apr}_{N(Q)}(A) \subseteq \underline{Apr}_{N(Q)}(\underline{Apr}_{N(Q)}(A))$. On the other hand, let $x \in \underline{Apr}_{N(Q)}(\underline{Apr}_{N(Q)}(A))$. Then $x \in q + N$ for some $q \in Q$ and $(q + N) \cap \underline{Apr}_{N(Q)}(A) \neq \emptyset$. Let $y \in (q + N) \cap \underline{Apr}_{N(Q)}(A)$. Then $y \in (q' + N) \subseteq A$ for some $q' \in Q$. Now $y \in (q + N) \cap (q' + N)$ implies $q = q'$. So $x \in (q' + N) \subseteq A$. Hence $x \in \underline{Apr}_{N(Q)}(A)$.
- viii) From (i), $\underline{Apr}_{N(Q)}(\overline{Apr}_{N(Q)}(A)) \subseteq \overline{Apr}_{N(Q)}(A)$. On the other hand, let $x \in \overline{Apr}_{N(Q)}(A)$. Then $x \in q + N$ for some $q \in Q$ and $(q + N) \cap A \neq \emptyset$. Now $(q + N) \cap A \neq \emptyset \Rightarrow q + N \subseteq \underline{Apr}_{N(Q)}(A) \Rightarrow x \in \underline{Apr}_{N(Q)}(\overline{Apr}_{N(Q)}(A))$. Thus, $\underline{Apr}_{N(Q)}(A) \subseteq \underline{Apr}_{N(Q)}(\overline{Apr}_{N(Q)}(A))$.
- ix) $x \in \underline{Apr}_{N(Q)}(A) \Leftrightarrow x \in q + N$ for some $q \in Q$ and $q + N \subseteq A \Leftrightarrow x \in q + N$ for some $q \in Q$ and $(q + N) \cap A^c = \emptyset \Leftrightarrow x \notin \overline{Apr}_{N(Q)}(A^c) \Leftrightarrow x \in (\overline{Apr}_{N(Q)}(A^c))^c$. Thus, $\underline{Apr}_{N(Q)}(A) = (\overline{Apr}_{N(Q)}(A^c))^c$.
- x) $x \in \overline{Apr}_{N(Q)}(A) \Leftrightarrow x \in q + N$ for some $q \in Q$ and $(q + N) \cap A \neq \emptyset \Leftrightarrow x \in q + N$ for some $q \in Q$ and $(q + N)$ is not a subset of $A^c \Leftrightarrow x \notin \underline{Apr}_{N(Q)}(A^c) \Leftrightarrow x \in (\underline{Apr}_{N(Q)}(A^c))^c$. Thus, $\overline{Apr}_{N(Q)}(A) = (\underline{Apr}_{N(Q)}(A^c))^c$.
- xi) $x \in \underline{Apr}_{N(Q)}(A \cap B) \Rightarrow x \in q + N$ for some $q \in Q$ and $q + N \subseteq A \cap B \Rightarrow x \in q + N$ for some $q \in Q$, $q + N \subseteq A$ and $q + N \subseteq B \Rightarrow x \in \underline{Apr}_{N(Q)}(A)$ and $x \in \underline{Apr}_{N(Q)}(B) \Rightarrow x \in \underline{Apr}_{N(Q)}(A) \cap \underline{Apr}_{N(Q)}(B)$. On the other hand, $x \in \underline{Apr}_{N(Q)}(A) \cap \underline{Apr}_{N(Q)}(B) \Rightarrow x \in q_1 + N$ for some $q_1 \in Q$, $q_1 + N \subseteq A$ and $x \in q_2 + N$ for some $q_2 \in Q$, $q_2 + N \subseteq B$. Now $x \in (q_1 + N) \cap (q_2 + N) \Rightarrow (q_1 + N) \cap (q_2 + N) \neq \emptyset \Rightarrow q_1 = q_2 \Rightarrow q_1 + N = q_2 + N \subseteq A \cap B$. Also, $x \in q_1 + N \Rightarrow x \in \underline{Apr}_{N(Q)}(A \cap B)$. Thus, $\underline{Apr}_{N(Q)}(A \cap B) = \underline{Apr}_{N(Q)}(A) \cap \underline{Apr}_{N(Q)}(B)$.
- xii) and xiii) Follows from (iv).
- xiv) $x \in \overline{Apr}_{N(Q)}(A \cup B) \Leftrightarrow x \in q + N$ for some $q \in Q$ and $(q + N) \cap (A \cup B) \neq \emptyset \Leftrightarrow x \in q + N$ for some $q \in Q$ and either $(q + N) \cap A \neq \emptyset$ or $(q + N) \cap B \neq \emptyset \Leftrightarrow x \in \overline{Apr}_{N(Q)}(A)$ or $x \in \overline{Apr}_{N(Q)}(B) \Leftrightarrow x \in \overline{Apr}_{N(Q)}(A) \cup \overline{Apr}_{N(Q)}(B)$. Thus, $\overline{Apr}_{N(Q)}(A \cup B) = \overline{Apr}_{N(Q)}(A) \cup \overline{Apr}_{N(Q)}(B)$.
- xv) Let $q \in Q$. Clearly $\underline{Apr}_{N(Q)}(q + N) \subseteq q + N \subseteq \overline{Apr}_{N(Q)}(q + N)$. Also $(q + N) \cap (q + N) \neq \emptyset$ and $q + N \subseteq q + N \Rightarrow \overline{Apr}_{N(Q)}(q + N) \subseteq q + N \subseteq \underline{Apr}_{N(Q)}(q + N)$.

Theorem 3.2 If N is a Q -subsemimodule of an R -semimodule M and K is a subsemimodule of M , then $\overline{Apr}_{N(Q)}(K)$ is a subsemimodule of M .

Proof. Let $x, y \in \overline{Apr}_{N(Q)}(K)$ and $r \in R$. Then $x \in q + N$, $y \in q' + N$ for some $q, q' \in Q$ and $(q + N) \cap K \neq \emptyset$, $(q' + N) \cap K \neq \emptyset$. Let $a \in (q + N) \cap K$ and $b \in (q' + N) \cap K$. Since K is a subsemimodule

of M , $a + b \in K$ and $ra \in K$. Now, $a \in q + N$ and $b \in q' + N$, implies $a + b \in (q + N) \oplus (q' + N) = q'' + N$ where $q'' \in Q$ is unique such that $q + q' + N \subseteq q'' + N$. Hence $(q'' + N) \cap N \neq \emptyset$. Hence $x + y \in \overline{Apr}_{N(Q)}(K)$. Similarly $rx \in \overline{Apr}_{N(Q)}(K)$. Therefore, $\overline{Apr}_{N(Q)}(K)$ is a subsemimodule of M .

The following example shows that the converse of the above theorem is not true.

Example 4. Consider the semiring $R = (\mathbb{Z}_0^+, +, \cdot)$, $M = \mathbb{Z}_0^+$ is an R -semimodule. Let $N = 4\mathbb{Z}_0^+$ and $K = \{2, 4\}$ be subsets of M . Here N is a Q -subsemimodule of M with $Q = \{0, 1, 2, 3\}$, $\overline{Apr}_{N(Q)}(K) (= 2\mathbb{Z}_0^+)$ is a subsemimodule of M but K is not a subsemimodule of M .

Theorem 3.3 Let N be a Q -subsemimodule of an R -semimodule M and K be a subsemimodule of M such that $N \subseteq K$. Then K is a subtractive extension of N if and only if $\underline{Apr}_{N(Q)}(K) = K$.

Proof. First suppose that K is a subtractive extension of N . Clearly, $K \subseteq \underline{Apr}_{N(Q)}(K)$. Now let $x \in \underline{Apr}_{N(Q)}(K)$. Then $x \in q + N$ for some $q \in Q$ and $(q + N) \cap K \neq \emptyset$. Let $a \in (q + N) \cap K$. Therefore, $a = q + n$ for some $n \in N \subseteq K$. Since $q + n = a \in K$ and K is a subtractive extension of N , $q \in K$. Thus, $x \in q + N \subseteq K$. Hence $\underline{Apr}_{N(Q)}(K) \subseteq K$. Thus, $\underline{Apr}_{N(Q)}(K) = K$. Conversely, suppose that $\underline{Apr}_{N(Q)}(K) = K$. Claim that K is a subtractive extension of N . Let $x \in N$, $x + y \in K$ for some $y \in M$. Now $y \in q' + N$ for some $q' \in Q$ and as $x \in N = q_0 + N$ implies $x + y \in (q_0 + N) \oplus (q' + N) = q' + N$. Hence $(q' + N) \cap K \neq \emptyset$. Now $y \in q' + N \Rightarrow y \in \underline{Apr}_{N(Q)}(K) = K$. Hence K is a subtractive extension of N .

Corollary 3.1 Let N be a Q -subsemimodule of an R -semimodule M . If K is a subtractive subsemimodule of M containing N , then $\underline{Apr}_{N(Q)}(K) = K$.

The following example shows that the converse of the above corollary is not true.

Example 5. Consider the semiring $R = (\mathbb{Z}_0^+, +, \cdot)$, $M = \mathbb{Z}_0^+$ is an R -semimodule. Let $N = \{0\}$ and $K = \langle 2, 3 \rangle = \mathbb{Z}_0^+ - \{1\}$ be subsemimodules of M . Here N is a Q -subsemimodule of M with $Q = \mathbb{Z}_0^+$, $\overline{Apr}_{N(Q)}(K) = K$ but K is not a subtractive subsemimodule of M .

Theorem 3.4 Let N be a Q -subsemimodule of an R -semimodule M . Then K is a subtractive extension of N if and only if $\underline{Apr}_{N(Q)}(K) = K$.

Proof. Let K be a subtractive extension of N . Clearly, $\underline{Apr}_{N(Q)}(K) \subseteq K$. Now $x \in K$. Then $x \in q + N$ for some $q \in Q$. Therefore, $x = q + n$ for some $n \in N$. Since K is a subtractive extension of N , $q \in K$. Therefore, $q + N \subseteq K$. Hence $x \in \underline{Apr}_{N(Q)}(K)$. Thus, $K \subseteq \underline{Apr}_{N(Q)}(K)$. Conversely, suppose that $\underline{Apr}_{N(Q)}(K) = K$. Claim that K is a subtractive extension of N . Let $x \in N$, $x + y \in K$ where $y \in M$. Now $y \in q' + N$ for some $q' \in Q$ and as

$x \in N = q_0 + N$ implies $x + y \in (q_0 + N) \oplus (q' + N) = q' + N$. Now $x + y \in K = \underline{Apr}_{N(Q)}(K)$ implies $(q' + N) \subseteq K$. Hence $y \in K$. Thus, K is a subtractive extension of N .

Corollary 3.2 Let N be a Q -subsemimodule of an R -semimodule M and K be a subtractive extension of N . Then $\underline{Apr}_{N(Q)}(K)$ is a subsemimodule of M .

The following example shows that the converse of the above corollary is not true.

Example 6. Consider the semiring $R = (\mathbb{Z}_0^+, +, \cdot)$, $M = \mathbb{Z}_0^+$ is an R -semimodule. Let $I = 4\mathbb{Z}_0^+$ and $K = \langle 2 \rangle \cup \{3\} = 2\mathbb{Z}_0^+ \cup \{3\}$ be subsets of M . Here, $\underline{Apr}_N(K) = 2\mathbb{Z}_0^+$ is a subsemimodule of M . But K is not a subtractive extension of N .

Corollary 3.3 Let N be a Q -subsemimodule of an R -semimodule M . If K is a subtractive subsemimodule of M containing N , then $\underline{Apr}_{N(Q)}(K) = K$.

Example 7. Consider the semiring $R = (\mathbb{Z}_0^+, +, \cdot)$, $M = \mathbb{Z}_0^+$ is an R -semimodule. Let $N = \{0\}$ and $K = \langle 2, 3 \rangle = \mathbb{Z}_0^+ - \{1\}$ be subsemimodules of M . Here N is a Q -subsemimodule of M with $Q = \mathbb{Z}_0^+$, $\underline{Apr}_N(K) = K$ but K is not a subtractive subsemimodule of M .

Theorem 3.5 Let $N \subseteq K$ be subsemimodules of an R -semimodule M and N a Q -subsemimodule of M . Then following statements are equivalent:

- 1) $\overline{Apr}_{N(Q)}(K) = K$;
- 2) $\underline{Apr}_{N(Q)}(K) = K$;
- 3) K is a subtractive extension of N ;
- 4) N is a $Q \cap K$ -ideal of K ;
- 5) $K/N_{(Q \cap K)}$ is a subsemimodule of an R -semimodule $M/N_{(Q)}$;
- 6) $K/N_{(Q \cap K)} \subseteq M/N_{(Q)}$.

Proof. Follows from [[12], Theorem 2.7] and Theorem 3.3, Theorem 3.4.

4. CONCLUSIONS

The present study extends rough set theory to semimodules over semirings by characterizing upper and lower approximations with respect to Bourne relations and partitioning subsemimodules. Conditions under which these approximations form subsemimodules, rough subsemimodules, and rough prime or

primary subsemimodules have been established, complemented by illustrative examples. These results provide a robust algebraic framework for further exploration of rough structures in semimodules over semirings.

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