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Research Paper

The Relation of Noncrossing Partitioning of Odd and Even Numbers with Catalan Numbers

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Abstract
Catalan numbers, denoted by C_n, are generally defined by the equation $C_n = \frac{1}{(n+1)}\binom{2n}{n}$ with n ≥ 0 and $n \in \mathbb{Z}$. Catalan numbers have forms that can be determined through general and recursive forms. Catalan numbers have several applications to various combinatorial problems, such as in recursive analysis and the application of combinatorial theory to partitions that can form Catalan numbers. The odd numbers are defined as integers that are not divisible by two, expressed in the form $\{2k + 1; k \in \mathbb{Z}\}$. Meanwhile, even numbers are defined as integers that are divisible by two, expressed in the form $\{2k; k \in \mathbb{Z}\}$. In this study we discuss the noncrossing partitions of positive odd numbers and positive even numbers. The results show those the noncrossing partitions have relationship with Catalan numbers.

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1. INTRODUCTION

In many counting problems, especially those involving recursively specified objects, a series of natural numbers known as "Catalan numbers" can be found. They bear the name Eugène Charles Catalan in honor of the Belgian mathematician. 1, 1, 2, 5, 14, 42, 132, 429, 1430, and so on are the first few Catalan numerals. A Catalan number is defined as a positive integer number obtained by calculating the combination structure of a row. Catalan numbers were discovered in 1844 by a Belgian mathematician, Eugene Charles Catalan. Catalan observed the number of valid or "well-formed parentheses" parentheses. The "well-formed parentheses" refers to a series of parentheses that are correctly nested, with a corresponding closing parenthesis for each opening parenthesis [1]. This implies that there must be an equal number of opening and closing parenthesis, and that their arrangement must not contravene the nesting constraints. Catalan number is denoted by C_n and id defined as $C_n = \frac{1}{(n+1)}\binom{2n}{n}$ [2]. Catalan number congruent to integers modulo. If m is a positive integer, then a is congruent to b modulo m ($a \equiv b(mod m)$) if m divides $(a - b)$. If m is not divide $(a - b)$ then a is not congruent to b modulo $m(a \not\equiv b(mod m))$ [3]. Some researchers investigate the Catalan numbers related with the modulo including the Catalan and Motzkin numbers [4], odd Catalan numbers modulo 2^k [5], and based on this research the Catalan prime numbers with rank k was investigated [3].

Koshy [2] and Stojadinovic [6] studied the triangulation of convex polygons and binary tresses in 2015, while Saracevic et al. [7] studied the combination of the Laticce Path based on Catalan numbers in 2018. Any binomial coefficient can be expressed as weighted sums along the Catalan triangle's rows, as demonstrated by Lee and Oh [8]. In 2020, a new conservative matrix that was derived using Catalan numbers [9], and a formula for creating different series, such as Catalan, Bernoulli, Harmonic, and Stirling numbers, was investigated by Boyadzhiev [10]. Furthermore, Catalan numbers are frequently used in many other domains, including medicine, engineering, computational geometry, geographic information systems, and cryptographic geodesy [11]. For data security, Saracevic et.al [12] employed dynamic programming-based Catalan keys for steganography in 2017. Catalan sequence also used for creating cipher in cryptography and it was a novelty because Catalan number is not used for encryption process [13,14]. Additionally, in 2019, Saracevic et.al [15] carried on investigating Catalan numbers and Dyck words for steganography. Ndagijimana [16] examines the characteristics and use of Catalan number in the secondary structure of RNA (ribonucleic acid). The pill problem is also solved using Catalan numbers [17]. Catalan number, Bell number, and Striling number ang their relationship with multiset is discussed [18]. In [19], the structure of a lattice of noncrossing partitions is examined. A key idea in combinatorics and related disciplines is this lattice, which is often referred to as the noncrossing partition lattice. For a more thorough explanation of Catalan number, see [2], [20], and [21].

This research aims to examine the application of Catalan numbers in the context of even and odd number noncrossing partitioning. Number partitioning is the process of dividing a number into smaller parts, and in this context, we will see how Catalan numbers can be used to analyze the structures and patterns that arise from partitioning even and odd numbers. The methods used in this study include recursive analysis and the application of combinatorial theory to identify the relationship between Catalan numbers and non-croosing partitioning results.

2. METHODS

2.1 Catalan Number

In the mathematical field of combinatorics, which examines the counting and arrangement of finite discrete structures, the Catalan numbers are among the most significant sequences. In [2] Catalan number C_n are ofently express as:

$$C_n = \frac{(2n)!}{(n+1)!n!} \tag{1}$$

$$C_n = \frac{1}{(n+1)}\binom{2n}{n}, n \geq 0 \tag{2}$$

Based on (1) or (2), we can form the sequence of numbers 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ... which is the sequence of Catalan numbers.

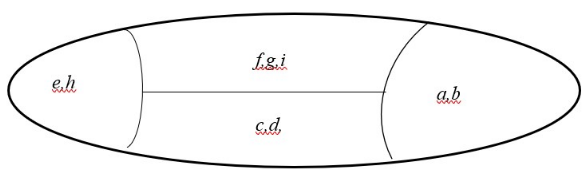

Figure 1. An Example of Partition

2.2 Partition

A family of subset A_i of a set A is a partition of A if each A_i is nonempty, the subsets are pairwise disjoint ($A_i \cap A_j = \emptyset$ if $i \neq j$), and the union of the subset is A ($\bigcup_{i \in I} A_i = A$, I is an index set). Every subset A_i is called as the block of the partition [2].

Figure 1 shows an example of partition. $A = \{a, b, c, d, e, f, g, h, i\}$. The partition of A is $P = \{\{a, b\}, \{c, d\}, \{e, h\}, \{f, g, i\}\}$. The blocks are $A_1 = \{a, b\}$, $A_2 = \{c, d\}$, $A_3 = \{e, h\}$, $A_4 = \{f, g, i\}$. To simplify we write P as $ab - cd - eh - fgi$.

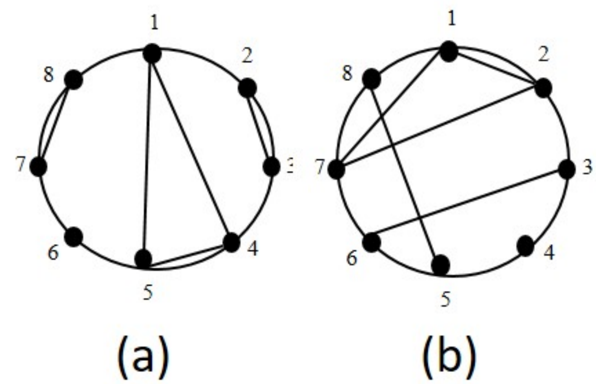

Figure 2. (a) 145-23-6-78 is a Noncrossing Partition; (b) 127-36-4-58 is not a Noncrossing Partition

2.3 Noncrossing Partition

A *noncrossing* partition π of the set $A = \{1, 2, 3, \ldots, n\}$ is the partition $\{a_1, a_2, a_3, \ldots, a_k\}$ of A so that if $p < q < r < s$, and $p, q \in \alpha_i$, and $r, s \in \alpha_j$, then $i = j$ [2].

A geometric description of the non-intersecting partition of the set $A = \{1, 2, 3, \ldots, n\}$ is also possible. Robert Steinberg of the University of California in Los Angeles, a National Academy of Sciences member and 1985 Leroy P. Steele Prize winner, is credited with this description [2]. For this purpose, consider a circle with n points on it, labelled 1 to n respectively. A noncrossing partition of A is a partition such that the convex hulls of the blocks are mutually exclusive.

Figure 2a and shows an example of a noncrossing partition of set $A = 1, 2, 3, 4, 5, 6, 7, 8$. The partition 145-23-6-78 is a noncrossing partition of A while in Figure 2b 127-36-4-58 is not a noncrossing partition.

Table 1. Example of Noncrossing Partition for 1 ≤ n ≤ 4 [2]

n	Noncrossing Partition	The Number
1	1	1
2	1-2 , 12	2
3	1-2-3 , 12-3 , 13-2, 1-23, 123	5
4	1-2-3-4, 12-3-4, 13-2-4, 14-2-3, 1-23-4, 1-24-3, 1-2-34, 12-34, 14-23 123-4, 124-3, 134-2, 1-234, 1234	14

The result of noncrossing partition also can be put in a table to simplify. Table 1 below shows an example of a noncrossing partition for 1 ≤ n ≤ 4 [2].

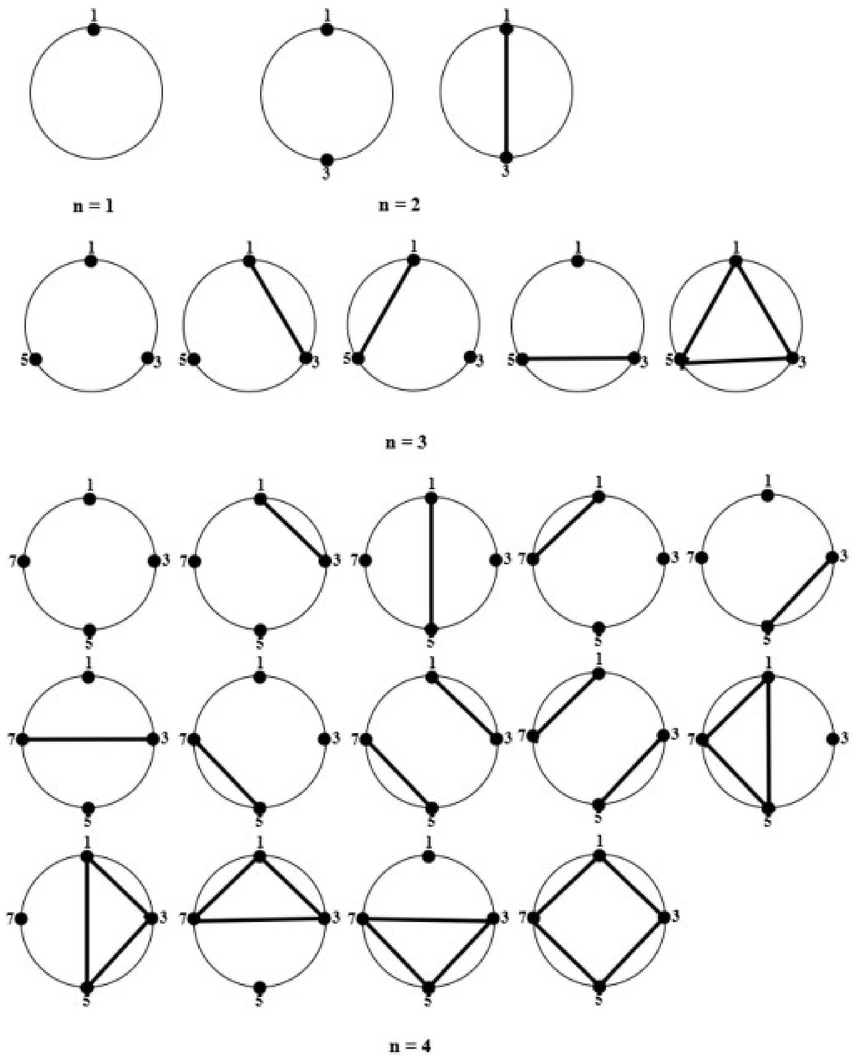

Figure 3. The Noncrossing Partition for $n = 4$, or for Odd Numbers 1, 3, 5, and 7

Table 2. Noncrossing Partition for $1 \leq n \leq 5$ or for Even Numbers 1, 3, 5, 7 and 9

n	Noncrossing Partition	The Number
1	1	1
2	1-3, 13	2
3	1-3-5, 13-5, 15-3, 1-35, 135	5
4	1-3-5-7 , 13-5-7, 15-3-7, 17-3-5, 35-1-7, 37-1-5, 57-1-5, 13-57, 17-35, 137-5, 135-7, 357-1, 157-3, 1357.	14
5	135-7-9, 137-5-9, 139-5-7, 157-3-9, 159-3-5, 179-3-5, 357-1-9, 359-1-7, 379-1-5, 579-1-3, 135-79, 139-57, 179-35 , 357-19, 579-13 1357-9, 3579-1, 1579-3, 1379-5, 1359-7 13-59-7, 13-57-9, 13-5-79, 17-3-79, 17-35-9, 19-35-7, 19-37-5, 19-3-57, 1-35-79, 1-39-57, 1-3-5-7-9, 13579	42

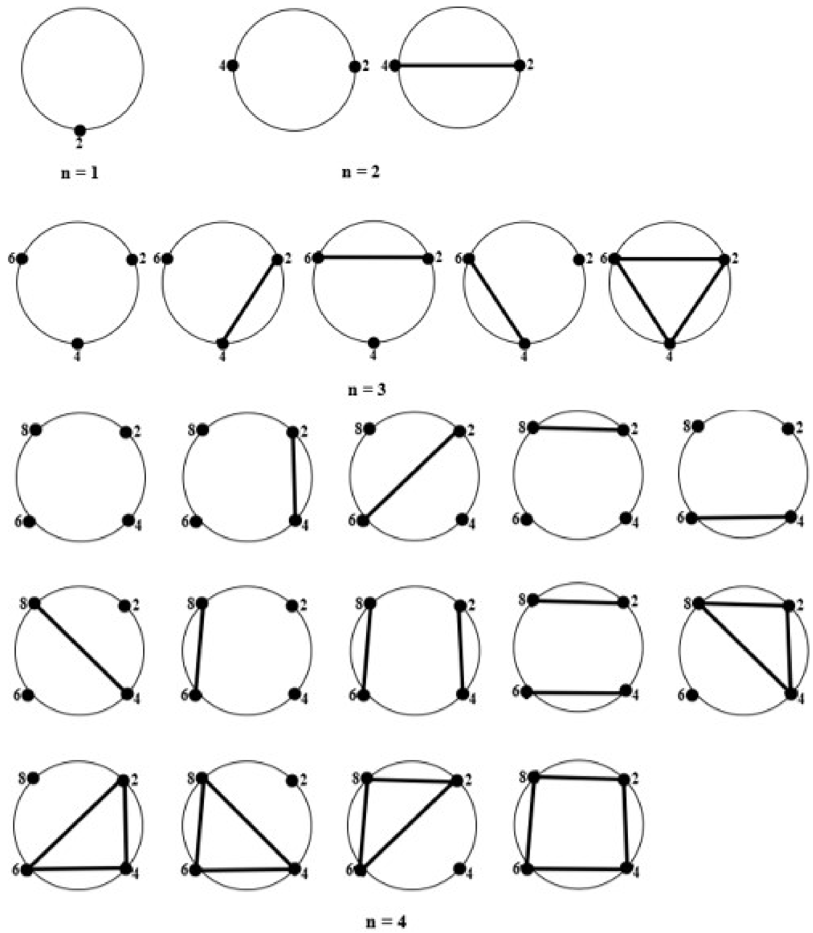

Figure 4. The Noncrossing Partition for $n = 4$, or for Even Numbers 2, 4, 6 and 8

Table 3. Noncrossing Partition for $1 \leq n \leq 5$ or for Even Numbers 2, 4, 6, 8, and 10

n	Noncrossing Partition	The Number
1	2	1
2	2-4, 24	2
3	2-4-6, 24-6, 26-4, 2-46, 246	5
4	2-4-6-8 , 24-6-8, 26-4-8, 28-4-6, 46-2-8, 48-2-6, 68-2-6, 24-68, 28-46, 248-6, 246-8, 468-2, 268-4, 2468.	14
5	24-6-8-10, 26-4-8-10, 28-4-6-10, 210-4-6-8, 46-8-10-2, 48-10-2-6, 410-2-6-8, 68-10-2-4, 610-2-4-8, 810-2-4-6, 246-8-10, 248-6-10, 2410-6-8, 268-4-10, 2610-4-6, 2810-4-6, 468-2-10, 4610-2-8, 4810-2-6, 6810-2-4, 246-810, 2410-68, 2810-46 , 468-210, 6810-24 2468-10, 46810-2, 26810-4, 24810-6, 24610-8 24-610-8, 24-68-10, 24-6-810, 26-4-810, 28-46-10, 210-46-8, 210-48-6, 210-4-68, 2-46-810, 2-410-68, 2-4-6-8-10, 246810	42

3. RESULTS AND DISCUSSION

3.1 Noncrossing Partition on Odd Number

In this section we give the results the noncrossing partition for the set of positive odd numbers $\{1, 3, 5, \ldots, 2n-1\}$. Due to the space limitation, we only give the results for $n = 5$, or for odd numbers 1, 3, 5, 7, and 9 in table form, and only for $n = 4$, or for odd numbers 1, 3, 5, and 7 for figure form.

Figure 3 shows the noncrossing partition for $n = 4$, or for odd numbers 1, 3, 5, and 7.

3.2 Noncrossing Partition on Even Number

Similar with the noncrossing partition for the set of odd numbers, in this section we do only give the results for $n = 5$, or for positive even numbers 2, 4, 6, 8, and 10 in table form, and only for $n = 4$, or for even numbers 2, 4, 6, and 8 for figure form.

Figure 4 shows the noncrossing partition for $n = 4$, or for even numbers 2, 4, 6 and 8.

As we already know that the sequence of Catalan number is 1, 2, 5, 14, 42, 132, 429, ... From the result of noncrossing partition for odd and even number the results show that the number of noncrossing partition (on the last column of Table 1 dan Table 2) are the same as the Catalan number for $n = 1, 2, 3, 4,$ and 5 which are 1, 2, 5, 14, and 42.

4. CONCLUSIONS

Based on the Results and Discussion above, we can conclude that the noncrossing partition on the first five positive odd numbers and the first five positive even numbers have relationship with Catalan numbers. The number of noncrossing partitions are the same as the Catalan numbers.

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