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Research Paper

Jordan Derivation on the Polynomial Ring *R[x]*

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Abstract
Given a ring R. An additive mapping $\delta{:}R\to R$ is called a Jordan derivation if δ satisfies $\delta(a^2) = \delta(a)a + a\delta(a)$, for every $a \in R$. Jordan derivation is one of the special forms of derivation. In this study, we investigate the Jordan derivation on the polynomial ring *R[x]* and investigate its properties. This study starts by constructing the Jordan derivation on the polynomial ring *R[x]*, followed by investigating its properties, including the relation between the Jordan derivation on the ring R and the polynomial ring *R[x]*. In addition, some concrete examples are presented to illustrate the main results obtained. This research is expected to contribute to further understanding of the properties of Jordan derivations on polynomial rings.

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1. INTRODUCTION

A ring is an algebraic structure defined as a set equipped with two binary operations that adhere to specific axioms. This structure plays a central role in numerous fields, including physics, chemistry, economics, finance, and cryptography, underscoring its wide-ranging relevance. One of the fundamental concepts in ring theory is the notion of a derivation, a function mapping a ring to itself that satisfies both linearity and the Leibniz rule. Interest in derivations across various algebraic frameworks continues to thrive. Numerous investigations have addressed derivations in different settings, including both rings and modules. For instance, Guven [1] analyzed special derivations in prime rings, followed by Golbasi and Koc [2] who extended this study to Lie ideals. Later, Ali et al. [3] focused on derivation commutativity in prime and semiprime rings, with subsequent generalizations by Ali and Alhazmi [4]. Atteya [5] further examined commutative properties of derivations in semiprime rings.In recent decades, research on derivations within ring structures has grown substantially, owing to their theoretical depth and diverse applications. One special form of derivation in ring theory is the Jordan derivation. This derivation has an important role in understanding the internal structure of a ring, especially in the context of ring expansion, ideals, as well as in the development of the concept of non-classical derivatives. The study of Jordan derivations not only provides new insights into the algebraic properties of rings, but also opens up opportunities for the application of this concept in various branches of mathematics, such as non-associative algebra, Lie theory, and functional analysis.

Additional contributions include Belkadi et al.'s work on nilpotent homoderivations [6] and n-Jordan homoderivations [7] in prime rings, and El-Sayiad et al.'s analysis of homoderivations in semiprimary rings [8]. El-Deken and El-Soufi [9] investigated derivation bindings, later evolving into homoderivation structures [10]. The research landscape remains dynamic, with Thomas et al. [11] studying derivations in various ring contexts, and Ezzat [12] delving into higher-order derivations. More recently, Gouda and Nabiel [13] expanded the concept to left derivations in 2024.

Within module theory, Bracic [14] examined derivations and their representations, while Gurjar and Patra [15] explored minimal generating sets for module derivations. The subject of commuting derivations has also attracted attention. Retert [16] analyzed them in simple rings, Chen and Wang [17] applied the concept to Lie algebras, and Maubach [18] proposed a conjecture on commuting derivations in ring settings. Pogudin [19] investigated commuting derivations in fields, whereas Fitriani et al. [20] examined both commuting and centralizing derivations in modules. Additionally, Fitriani et al. [21] explored f-derivations in polynomial modules, highlighting how ring derivations serve as the basis for more complex module derivation structures.

Although Jordan derivations have been widely studied, especially in the contexts of commutative, non-commutative, and semi-simple rings, their behavior in polynomial rings $R[x]$ has received relatively limited attention. This article seeks to fill that gap by examining the relationship between Jordan derivations on a general ring R and those on the polynomial ring $R[x]$. In addition, it aims to investigate the structural properties of such derivations in $R[x]$, thereby contributing to a deeper understanding of Jordan derivations within extended ring frameworks. The present results may further be developed to define derivations on the rough quotient module as constructed by Adelia et al. [22].

2. METHODS

The type of research used by the author is literature study, which involves collecting and processing research materials based on related references such as journals, books, and articles pertinent to this study.

3. RESULTS AND DISCUSSION

3.1 Jordan Derivation on the Ring R

A derivation on the ring R is an additive operation that satisfies the Leibniz rule. This derivation can be applied to the elements of the ring, including those in the polynomial ring *R[x]*, where each polynomial derivative follows a pattern similar to differentiation in calculus.

Theorem 5. Every derivation on a ring is a Jordan derivation.
Proof. Given a ring R and a map $d : R \to R$, let d be a derivation if it satisfies the following two properties:

i) $d(a+b)=d(a)+d(b)$, for all $a, b \in R$, and
ii) $d(ab)=d(a)b+ad(b)$, for all $a, b \in R$.

From the definition of derivation, it follows that:

$$d(a^2) = d(aa).$$

Since d is a derivation, the property $d(ab)=d(a)b+ad(b)$ holds. By substituting $b=a$, we obtain:

$$d(a^2) = d(a)a + ad(a).$$

Since $d(a^2) = d(a)a + ad(a)$ holds for all $a \in R$, it follows that d satisfies the definition of a Jordan derivation. Thus, it is proven that every derivation on a ring is a Jordan derivation.

Example 3. Given the ring $R = M_2(\mathbb{Z}_2)$, the function $d : R \to R$ is defined as:

$$d(A) = A^T - A,$$

where A^T is the transpose of the matrix A.

It will be shown that d is a Jordan derivation.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}_2)$ be arbitrary. We have:

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix}.$$

The transpose of A^2 is:

$$(A^2)^T = \begin{bmatrix} a^2 + bc & ac + dc \\ ab + bd & bc + d^2 \end{bmatrix}.$$

Thus, we obtain:

$$\begin{aligned} d(A^2) &= (A^2)^T - A^2 \\ &= \begin{bmatrix} a^2 + bc & ac + dc \\ ab + bd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & ac + dc - ab - bd \\ ab + bd - ac - dc & 0 \end{bmatrix}. \end{aligned}$$

On the other hand,

$$d(A) = A^T - A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & c - b \\ b - c & 0 \end{bmatrix}.$$

Now, calculate:

$$\begin{aligned} d(A)A + Ad(A) = &\begin{bmatrix} 0 & c - b \\ b - c & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &+ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & c - b \\ b - c & 0 \end{bmatrix} \end{aligned}$$

This results in:

$$\begin{aligned} &= \begin{bmatrix} c^2 - bc & cd - bd \\ ab - ac & b^2 - bc \end{bmatrix} + \begin{bmatrix} b^2 - bc & ac - ab \\ bd - cd & c^2 - bc \end{bmatrix} \\ &= \begin{bmatrix} c^2 - bc + b^2 - bc & cd - bd + ac - ab \\ ab - ac + bd - cd & b^2 - bc + c^2 - bc \end{bmatrix}. \end{aligned}$$

Simplifying further:

$$= \begin{bmatrix} 0 & ac + dc - ab - bd \\ ab + bd - ac - dc & 0 \end{bmatrix}.$$

Thus, $d(A^2) = d(A)A + Ad(A)$. Therefore, it is proven that the function d is a Jordan derivation on the ring $M_2(\mathbb{Z}_2)$.

Next, it will be investigated whether the function d is a derivation.
Let arbitrary matrices A and B be given as follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

We calculate:

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$

The transpose of AB is:

$$(AB)^T = \begin{bmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{bmatrix}.$$

Thus, we obtain:

$$\begin{aligned} d(AB) &= (AB)^T - AB \\ &= \begin{bmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{bmatrix} - \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \\ &= \begin{bmatrix} 0 & ce-af \\ af-ce & 0 \end{bmatrix}. \end{aligned}$$

On the other hand, the transpose of A is:

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad d(A) = A^T - A = \begin{bmatrix} 0 & c-b \\ b-c & 0 \end{bmatrix}.$$

The transpose of B is:

$$B^T = \begin{bmatrix} e & g \\ f & h \end{bmatrix}, \quad d(B) = B^T - B = \begin{bmatrix} 0 & g-f \\ f-g & 0 \end{bmatrix}.$$

We now calculate

$$\begin{aligned} d(A)B + Ad(B) &= \begin{bmatrix} 0 & c-b \\ b-c & 0 \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &\quad + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & g-f \\ f-g & 0 \end{bmatrix} \\ &= \begin{bmatrix} (c-b)g & (c-b)h \\ (b-c)e & (b-c)f \end{bmatrix} \\ &\quad + \begin{bmatrix} b(f-g) & a(g-f) \\ d(f-g) & c(g-f) \end{bmatrix} \\ &= \begin{bmatrix} (c-b)g + b(f-g) & (c-b)h + a(g-f) \\ (b-c)e + d(f-g) & (b-c)f + c(g-f) \end{bmatrix} \\ &= \begin{bmatrix} cg - bg + bf - bg & ch - bh + ag - af \\ be - ce + df - dg & bf - cf + cg - cf \end{bmatrix}. \end{aligned}$$

For example, if we choose

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

we get:

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Thus, we compute:

$$\begin{aligned} d(AB) &= (AB)^T - AB \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^T - \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

On the other hand:

$$\begin{aligned} d(A) = A^T - A &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^T - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned}$$

Similarly:

$$\begin{aligned} d(B) = B^T - B &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^T - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

We obtain:

$$\begin{aligned} d(A)B + Ad(B) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}. \end{aligned}$$

Thus, $d(AB) \neq d(A)B + Ad(B)$, so the function d is not a derivation.

3.2 Properties of Jordan Derivations

The linear combination of Jordan derivations is a concept that arises in algebra theory, particularly within ring structures. By preserving the fundamental properties of Jordan derivations, this linear combination forms a new derivation that still adheres to the rules applicable to Jordan derivations.

Theorem 6. Given a ring R, if d_1 and d_2 are Jordan derivations on R, then $d = c_1 d_1 + c_2 d_2$, where c_1 and c_2 are elements in the ring R, is a Jordan derivation on R.

Proof. Given two Jordan derivations d_1 and d_2 on a ring R defined as follows:

$$d_1(a^2) = d_1(a)a + ad_1(a)$$

and

$$d_2(a^2) = d_2(a)a + ad_2(a)$$

for every $a \in R$.

It will be proven that the linear combination d is also a Jordan derivation.

$$\begin{aligned} d(a^2) &= c_1 d_1(a^2) + c_2 d_2(a^2) \\ &= c_1 \left(d_1(a)a + ad_1(a)\right) + c_2 \left(d_2(a)a + ad_2(a)\right) \\ &= \left(c_1 d_1(a) + c_2 d_2(a)\right) a + a \left(c_1 d_1(a) + c_2 d_2(a)\right) \\ &= d(a)a + ad(a). \end{aligned}$$

Since $d(a^2) = d(a)a + ad(a)$, this shows that the linear combination $d = c_1d_1 + c_2d_2$ satisfies the Jordan derivation equation. Therefore, the linear combination of two Jordan derivations is also a Jordan derivation.

Theorem 7. Given a ring $R, if\, d_1, d_2, \ldots, d_n$ are Jordan derivations on R, then the linear combination $d = c_1d_1 + c_2d_2 + ... + c_nd_n$, where $c_1, c_2, \ldots, c_n$ are elements in the ring R, is a Jordan derivation on R.

Proof. Let d be defined as a linear combination of n Jordan derivations, i.e.,

$$d = c_1d_1 + c_2d_2 + ... + c_nd_n, \text{where } c_1, c_2, \ldots, c_n \in R.$$

It will be proven that d is also a Jordan derivation.

$$\begin{aligned} d(a^2) &= (c_1d_1 + c_2d_2 + \cdots + c_nd_n)(a^2) \\ &= c_1d_1(a^2) + c_2d_2(a^2) + \cdots + c_nd_n(a^2) \\ &= c_1\big(d_1(a)a + ad_1(a)\big) + c_2\big(d_2(a)a + ad_2(a)\big) \\ &\quad + \cdots + c_n\big(d_n(a)a + ad_n(a)\big) \\ &= \big(c_1d_1(a) + c_2d_2(a) + \cdots + c_nd_n(a)\big)a \\ &\quad + a\big(c_1d_1(a) + c_2d_2(a) + \cdots + c_nd_n(a)\big) \\ &= d(a)a + ad(a). \end{aligned}$$

Therefore, d is also a Jordan derivation on R.

If R is a ring and d_1, d_2 are two Jordan derivations, then $d_1 \circ d_2$ is not necessarily a Jordan derivation. An example is provided below.

Example 4. Let $R = M_2(\mathbb{R})$, the ring of 2×2 matrices with real entries. Two Jordan derivations are defined as follows:

$$d_1(A) = A^T, \qquad d_2(A) = EA - AE$$

Let E and A be arbitrary matrices given by:

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then, we obtain:

$$EA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}, AE = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}.$$

$$d_2(A) = EA - AE = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix}.$$

Since $d_1(A) = A^T$,

$$d_1(d_2(A)) = \left(\begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix}\right)^T = \begin{bmatrix} c & 0 \\ d-a & -c \end{bmatrix}.$$

The transpose of A is:

$$d_1(A) = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

$$\begin{aligned} d_2(d_1(A)) &= E(d_1(A)) - (d_1(A))E \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} - \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} b & d \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \\ &= \begin{bmatrix} b & d-a \\ 0 & -b \end{bmatrix}. \end{aligned}$$

From these calculations, we obtain the following results:

$$d_1(d_2(A)) = \begin{bmatrix} c & 0 \\ d-a & -c \end{bmatrix}, \quad d_2(d_1(A)) = \begin{bmatrix} b & d-a \\ 0 & -b \end{bmatrix}.$$

Therefore, $\mathbf{d_1(d_2(A)) \neq d_2(d_1(A))}$. Hence, the composition of the two Jordan derivations $d_1(A) = A^T$ and $d_2(A) = EA - AE$ is not a Jordan derivation.

3.3 Jordan derivation on the polynomial ring *R[x]*

In the polynomial ring *R[x]*, a Jordan derivation is defined as a mapping that satisfies a specific rule. Unlike ordinary derivations, which follow the Leibniz rule, Jordan derivations in the polynomial ring adhere to a special rule concerning the square of elements, as explained in the following definition.

Definition 8. An additive mapping $d : R[x] \to R[x]$ with the variable x and coefficients in a commutative ring R is called a Jordan derivation if it satisfies the condition:

$$d(p(x)^2 = d(p(x))p(x) + p(x)d(p(x)), \text{for every } p(x) \in R[x].$$

Theorem 9. Given a ring R, if $d : R \to R$ is a Jordan derivation, then there exists $\hat{d} : R[x] \to R[x]$ that is a Jordan derivation in $R[x]$.

Proof. It is defined as

$$\hat{d}\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} d(a_i)x^i \in R[x].$$

It will be shown that $\hat{d}$ is a Jordan derivation.

Given any $p(x) \in R[x]$, we have:

$$p(x) = \sum_{i=0}^{n} a_i x^i,$$

where $a_i \in R$.

It follows that

$$\begin{aligned} p(x)^2 &= \left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{n} a_j x^j\right) \\ &= \sum_{k=0}^{2n}\left(\sum_{\substack{i+j=k \\ i,j\le n}} a_i a_j\right)x^k. \end{aligned}$$

$$\hat{d}(p(x)^2) = \sum_{k=0}^{2n} d\left(\sum_{\substack{i+j=k\\ i,j\leq n}} a_i a_j\right) x^k.$$

Since d is a Jordan derivation on R, it satisfies

$$d(a_i a_j) = d(a_i)a_j + a_i d(a_j),$$

for all $a_i, a_j \in R$.

Consequently,

$$d\left(\sum_{\substack{i,j=0\\ i+j=k}}^{n} a_i a_j\right) = \sum_{\substack{i,j=0\\ i+j=k}}^{n} d(a_i a_j) = \sum_{\substack{i,j=0\\ i+j=k}}^{n} \left(d(a_i)a_j + a_i d(a_j)\right).$$

Thus,

$$\hat{d}(p(x)^2) = \sum_{k=0}^{2n} \left(\sum_{\substack{i,j=0\\ i+j=k}}^{n} \left(d(a_i)a_j + a_i d(a_j)\right)\right) x^k.$$

By the definition of $\hat{d}$, it follows that

$$\hat{d}(p(x)) = \sum_{i=0}^{n} d(a_i)x^i.$$

On the other hand,

$$\begin{aligned}
&\hat{d}(p(x))p(x) + p(x)\hat{d}(p(x))\\
&= \left(\sum_{i=0}^{n} d(a_i)x^i\right)\left(\sum_{j=0}^{n} a_j x^j\right) + \left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{n} d(a_j)x^j\right)\\
&= \sum_{k=0}^{2n}\left(\sum_{\substack{i,j=0\\ i+j=k}}^{n} d(a_i)a_j\right)x^k + \sum_{k=0}^{2n}\left(\sum_{\substack{i,j=0\\ i+j=k}}^{n} a_i d(a_j)\right)x^k\\
&= \sum_{k=0}^{2n}\left(\sum_{\substack{i,j=0\\ i+j=k}}^{n} \left(d(a_i)a_j + a_i d(a_j)\right)\right)x^k.
\end{aligned}$$

Since $\hat{d}(p(x)^2) = \hat{d}(p(x))p(x) + p(x)\hat{d}(p(x))$, it is proved that $\hat{d}$ a Jordan derivation.

Example 5. Given that R is a commutative ring and $R[x]$ is the polynomial ring with variable x and coefficients in R, a mapping $d : R[x] \to R[x]$ is defined as follows:

$$d(f(x)) = af'(x),$$

where $a \in R$.

It will be shown that the mapping d satisfies the Jordan derivation property.

Based on the definition of d, the following is obtained:

$$d(f(x)^2) = a \cdot \frac{d}{dx}(f(x)^2) = a \cdot (2f(x)f'(x)) = 2af(x)f'(x).$$

From the definition $d(f(x)) = af'(x)$, it follows that:

$$\begin{aligned}
d(f(x))f(x) &= (af'(x))f(x) = af'(x)f(x),\\
f(x)d(f(x)) &= f(x)(af'(x)) = af'(x)f(x),\\
d(f(x))f(x) + f(x)d(f(x)) &= af'(x)f(x) + af'(x)f(x)\\
&= 2af(x)f'(x).
\end{aligned}$$

Since $d(f(x)^2) = d(f(x))f(x) + f(x)d(f(x))$, it is proven that d is a Jordan derivation.

Theorem 10. Given that d_1 and d_2 are Jordan derivations on $R[x]$, then $d = c_1d_1 + c_2d_2$, where $c_1, c_2 \in R$, is also a Jordan derivation on $R[x]$.

Proof. Given that d_1 and d_2 are Jordan derivations on $R[x]$, they satisfy the following properties:

$$\begin{aligned}
d_1(f(x)^2) &= d_1(f(x))f(x) + f(x)d_1(f(x)),\\
d_2(f(x)^2) &= d_2(f(x))f(x) + f(x)d_2(f(x)).
\end{aligned}$$

Define $d = c_1d_1 + c_2d_2$ with $c_1, c_2 \in R$. It will be shown that d is also a Jordan derivation on $R[x]$.

Substituting both equations into $d(f(x)^2)$ gives:

$$\begin{aligned}
d(f(x)^2) &= c_1d_1(f(x)^2) + c_2d_2(f(x)^2)\\
&= c_1\left(d_1(f(x))f(x) + f(x)d_1(f(x))\right)\\
&\quad + c_2\left(d_2(f(x))f(x) + f(x)d_2(f(x))\right)\\
&= c_1d_1(f(x))f(x) + c_1f(x)d_1(f(x))\\
&\quad + c_2d_2(f(x))f(x) + c_2f(x)d_2(f(x))\\
&= \left(c_1d_1(f(x)) + c_2d_2(f(x))\right)f(x)\\
&\quad + f(x)\left(c_1d_1(f(x)) + c_2d_2(f(x))\right).
\end{aligned}$$

Since $d(f(x)) = c_1d_1(f(x)) + c_2d_2(f(x))$, it follows that

$$d(f(x)^2) = d(f(x))f(x) + f(x)d(f(x)).$$

Thus, the linear combination $d = c_1d_1 + c_2d_2$ satisfies the Jordan derivation property on $R[x]$.

Example 6. Given a ring $R = \mathbb{R}$, which is the set of real numbers, and $R[x]$, the polynomial ring with coefficients in R, two Jordan derivations d_1 and d_2 on $R[x]$ are defined as follows:

$$\begin{aligned}
d_1(f(x)) &= f'(x),\\
d_2(f(x)) &= xf'(x).
\end{aligned}$$

Next, the linear combination of the Jordan derivations d_1 and d_2 is defined as:

$$d(f(x)) = 2d_1(f(x)) + 3d_2(f(x)) = 2f'(x) + 3xf'(x),$$

with coefficients $2, 3 \in \mathbb{R}$.

To prove that d is a Jordan derivation, it must satisfy the condition:

$$d(f(x)^2) = d(f(x))f(x) + f(x)d(f(x))$$

for every $f(x) \in R[x]$.

$$d(f(x)^2) = 4f(x)f'(x) + 6xf(x)f'(x).$$

On the other hand,

$$\begin{aligned} d(f(x))f(x) + f(x)d(f(x)) &= (2f'(x) + 3xf'(x))f(x) \\ &\quad + f(x)(2f'(x) + 3xf'(x)) \\ &= 2f'(x)f(x) + 3xf'(x)f(x) \\ &\quad + 2f'(x)f(x) + 3xf'(x)f(x) \\ &= 4f(x)f'(x) + 6xf(x)f'(x). \end{aligned}$$

Thus, it follows that

$$d(f(x)^2) = d(f(x))f(x) + f(x)d(f(x)).$$

Therefore, the linear combination $d = c_1 d_1 + c_2 d_2$, with $d_1(f(x)) = f'(x)$ and $d_2(f(x)) = xf'(x)$, satisfies the Jordan derivation property on $R[x]$, as it fulfills the condition:

$$d(f(x)^2) = d(f(x))f(x) + f(x)d(f(x)).$$

This confirms that d is a Jordan derivation on the polynomial ring $R[x]$.

If R is a ring and d_1, d_2 are two Jordan derivations on $R[x]$, then the composition $d_1 \circ d_2$ is not necessarily a Jordan derivation. The following example illustrates this.

Example 7. Given a ring $R = \mathbb{R}[x]$ and the function $f(x) = x^3$, the Jordan derivations d_1 and d_2 are defined as follows:

$$\begin{aligned} d_1(f(x)) &= f'(x), \\ d_2(f(x)) &= xf'(x). \end{aligned}$$

It will be shown that $d_1 \circ d_2$ is not a Jordan derivation on the polynomial ring $R[x]$ as follows:

$$d_2(f(x)) = x^2 f'(x).$$

Since $f(x) = x^3$, its derivative is $f'(x) = 3x^2$, thus:

$$\begin{aligned} d_2(f(x)) &= x^2 \cdot 3x^2 = 3x^4, \\ d_1\left(d_2(f(x))\right) &= d_1(3x^4) = 12x^3. \end{aligned}$$

Next, $d_2\left(d_1(f(x))\right)$ is computed as follows:

$$d_1(f(x)) = f'(x).$$

Since $f'(x) = 3x^2$, it follows that:

$$\begin{aligned} d_1(f(x)) &= 3x^2, \\ d_2\left(d_1(f(x))\right) &= d_2(3x^2) = x^2 \cdot 3x^2 = 3x^4. \end{aligned}$$

Since $\mathbf{d_1}\left(\mathbf{d_2}(\mathbf{f}(\mathbf{x}))\right) = \mathbf{12x^3}$ and $\mathbf{d_2}\left(\mathbf{d_1}(\mathbf{f}(\mathbf{x}))\right) = \mathbf{3x^4}$, the composition of Jordan derivations is not a Jordan derivation.

4. CONCLUSIONS

The Jordan derivation applied to square elements in a ring will produce a consistent pattern of changes, which is often used to form and study the ideal structure within the ring. It can be concluded that the linear combination of Jordan derivations on a ring is also a Jordan derivation. Furthermore, if a Jordan derivation exists on a ring R, then this mapping can be extended to a Jordan derivation on $R[x]$. Additionally, the linear combination of Jordan derivations remains a Jordan derivation on $R[x]$. In particular, the composition of two Jordan derivations does not always result in a Jordan derivation. Moreover, every derivation on a ring is always a Jordan derivation, but not every Jordan derivation is a derivation. A concrete example in the matrix ring $M_2(\mathbb{Z}_2)$ with the function $d(A) = A^T - A$ demonstrates that this mapping satisfies the properties of a Jordan derivation but does not always fulfill the Leibniz rule, making it a Jordan derivation that is not a derivation.

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