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Research Paper

On the Construction of Rough Quotient Modules in Finite Approximation Spaces

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Abstract

Let $\mathcal{S}$ be a set and φ an equivalence relation on $\mathcal{S}$. The pair $(\mathcal{S}, \varphi)$ forms an approximation space, where the relation φ partitions $\mathcal{S}$ into mutually disjoint equivalence classes. For any subset $B' \subseteq \mathcal{S}$, the lower approximation $\underline{Apr}(B')$ is defined as the union of all equivalence classes entirely contained in B', while the upper approximation $\overline{Apr}(B')$ is the union of all equivalence classes that have a non-empty intersection with B'. The subset B' is called a rough set in $(\mathcal{S}, \varphi)$ if $\underline{Apr}(B') \neq \overline{Apr}(B')$. If, in addition, B' satisfies certain algebraic conditions, it is termed a rough module. This paper investigates the construction of rough quotient rings and rough quotient modules within such approximation spaces. The approach is developed using finite sets to facilitate the algebraic formulation and analysis of these rough structures.

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1. INTRODUCTION

Rough set theory, introduced by Pawlak, is a mathematical framework developed to manage vagueness and uncertainty, particularly within information systems, data analysis, machine learning, and decision support systems. Since its inception, the theory has been widely studied and applied across numerous disciplines. From an algebraic perspective, its foundational concepts have been generalized and extended to broader mathematical structures, including non-classical logics, ordered systems, and classical algebraic frameworks such as groups, rings, and modules.

Bagismaz and Ozcan [1] pioneered the notion of rough semigroups within the framework of approximation spaces, establishing a foundation for exploring algebraic structures under uncertainty. Building on this foundation, Neelima and Isaac [2] studied anti-homomorphisms in rough groups, shedding light on the structural behavior of group elements under rough approximations. Wang and Chen [3] further contributed to this area by analyzing properties of rough groups and demonstrating their relevance to computer science applications, including data mining and artificial intelligence.

Sinha and Prakash extended these ideas to module theory by investigating structural properties of rough projective modules [4], and subsequently introducing the concept of rough injective modules [5], thereby offering a dual perspective and advancing the development of a rough module theory analogous to the classical theory over rings.

Earlier, Miao et al. [6] had conducted foundational research on rough groups, subgroups, and homomorphisms, while Zhang et al. [7] explored rough modules and quotient modules over rough rings. Later, Isaac and Paul [8] introduced the concept of rough G-modules and studied their structural properties. Nugraha et al. [9] examined applications of rough set theory to group structures, followed by Hafifulloh et al. [10], who studied properties of rough V-coexact rows in rough groups. Parallel investigations into rough rings were carried out by Agusfrianto et al. [11], who explored rough rings, rough subrings, and rough ideals. This was extended by Agusfrianto and Ambarwati [12], who characterized properties of rough ideals in rough rings.

Yanti et al. [13] applied rough set theory to projective modules, forming the basis for rough projective modules and their associated properties.

The study of rough modules over rings was initiated by Davvaz and Mahdavipour [14], and further advanced by Zhang et al. [7], Chen et al. [15], and Vijayabalaji et al. [16]. Recent developments in 2024 include the work of Agusfrianto et al. [17], who introduced rough bimodules, and Dwiyanti et al. [18], who developed the concept of X-sub-linearly independent elements in rough groups. Fitriani et al. [19] proposed the rough X-sub exact sequence for rough modules over rough rings, Meyer introduced soft and rough module [20], while Rahmawati et al. [21] conducted a detailed investigation into rough submodules.

Despite the rich body of literature on rough algebraic structures, no study to date has specifically addressed the construction and characterization of rough quotient modules over finite sets. This research aims to fill that gap by characterizing rough quotient modules over rough rings.

Therefore, we begin by presenting the definition of rough quotient rings, accompanied by illustrative examples involving finite sets and rough ideals. We then establish approximation spaces for finite groups, rings, and modules, and provide concrete examples to elucidate these structures.

2. METHODS

This study employs a theoretical approach within the framework of abstract algebra and rough set theory to characterize rough quotient modules over rough rings on finite sets. It begins with basic definitions of modules, rough rings, and lower and upper approximations in finite approximation spaces. Rough quotient modules are constructed using suitable equivalence relations. We also provide examples to illustrate rough module constructions, with proofs carried out using algebraic methods and rough set-theoretic reasoning.

3. RESULTS AND DISCUSSION

Given a rough ring R with operations $+$ and $\cdot$, and a rough ideal V of R. Let $R/V = \{\overline{p} = p + \overline{Apr}(V) \mid p \in R\}$. We define addition and multiplication on R/V as follows: $\overline{p_1} + \overline{p_2} = (p_1 + \overline{Apr}(V)) + (p_2 + \overline{Apr}(V)) = (p_1 + p_2) + \overline{Apr}(V) = \overline{p_1 + p_2}$ and $\overline{p_1} \cdot \overline{p_2} = (p_1 + \overline{Apr}(V))(p_2 + \overline{Apr}(V)) = (p_1 p_2) + \overline{Apr}(V) = \overline{p_1 \cdot p_2}$, for every $p_1 + \overline{Apr}(V), p_2 + \overline{Apr}(V) \in R/V$. We will show that $\langle R/V, +, \cdot \rangle$ forms a rough ring:

1. Let $\overline{p_1}, \overline{p_2} \in R/V$. We have $\overline{p_1} + \overline{p_2} = \overline{p_1 + p_2} \in R/V$.
2. Let $\overline{p_1}, \overline{p_2}$, and $\overline{p_3} \in R/V$, $\overline{p_1} + (\overline{p_2} + \overline{p_3}) = \overline{p_1} + \overline{(p_2 + p_3)} = \overline{p_1 + (p_2 + p_3)} = \overline{(p_1 + p_2) + p_3} = \overline{p_1 + p_2} + \overline{p_3} = (\overline{p_1} + \overline{p_2}) + \overline{p_3}$.
3. There is $\overline{0} \in R/V$ such that $\overline{p_1} + \overline{0} = \overline{p_1 + 0} = \overline{p_1}$ and $\overline{0} + \overline{p_1} = \overline{0 + p_1} = \overline{p_1}$, for every $\overline{p_1} \in R/V$.
4. We can see from Table 2, there is $\overline{-p_1} \in R/V$ such that $\overline{p_1} + \overline{-p_1} = \overline{p_1 + (-p_1)} = \overline{0}$ and $\overline{-p_1} + \overline{p_1} = \overline{(-p_1) + p_1} = \overline{0}$.
5. For every $\overline{p_1}, \overline{p_2} \in R/V$, $\overline{p_1} + \overline{p_2} = \overline{p_1 + p_2} = \overline{p_2 + p_1} = \overline{p_2} + \overline{p_1}$.
6. For every $\overline{p_1}, \overline{p_2} \in R/V$, $\overline{p_1} \cdot \overline{p_2} = \overline{p_1 \cdot p_2} \in R/V$.
7. For every $\overline{p_1}, \overline{p_2}, \overline{p_3} \in R/V$, $\overline{p_1} \cdot (\overline{p_2} \cdot \overline{p_3}) = \overline{p_1} \cdot \overline{(p_2 \cdot p_3)} = \overline{p_1 \cdot (p_2 \cdot p_3)} = \overline{(p_1 \cdot p_2) \cdot p_3} = \overline{(p_1 \cdot p_2)} \cdot \overline{p_3} = (\overline{p_1} \cdot \overline{p_2}) \cdot \overline{p_3}$.
8. For every $\overline{p_1}, \overline{p_2}, \overline{p_3} \in R/V$, $\overline{p_1} \cdot (\overline{p_2} + \overline{p_3}) = \overline{p_1} \cdot \overline{p_2 + p_3} = \overline{p_1 \cdot (p_2 + p_3)} = \overline{p_1 \cdot p_2 + p_1 \cdot p_3} = \overline{p_1 \cdot p_2} + \overline{p_1 \cdot p_3} = \overline{p_1} \cdot \overline{p_2} + \overline{p_1} \cdot \overline{p_3}$ and $(\overline{p_1} + \overline{p_2}) \cdot \overline{p_3} = \overline{p_1 + p_2} \cdot \overline{p_3} = \overline{(p_1 + p_2) \cdot p_3} = \overline{p_1 \cdot p_3 + p_2 \cdot p_3} = \overline{p_1 \cdot p_3} + \overline{p_2 \cdot p_3} = \overline{p_1} \cdot \overline{p_3} + \overline{p_2} \cdot \overline{p_3}$.

It is proven that $\langle R/V, +, \cdot \rangle$ is a rough ring, called a rough quotient ring.

Example 1. *Let $(\mathcal{S}, \varphi)$ be an approximation space with $\mathcal{S} = \mathbb{Z}_{48}$. We define an equivalence relation φ such that $c_1 \varphi c_2$ holds if and only if $c_1 -_{48} c_2 = 4l$ with $c_1, c_2 \in \mathcal{S}$ and $l \in \mathbb{Z}$. The equivalence classes on the set $\mathcal{S}$ are obtained as follows:*

$$Q_1 = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}, \overline{24}, \overline{28}, \overline{32}, \overline{36}, \overline{40}, \overline{44}\};$$
$$Q_2 = \{\overline{1}, \overline{5}, \overline{9}, \overline{13}, \overline{17}, \overline{21}, \overline{25}, \overline{29}, \overline{33}, \overline{37}, \overline{41}, \overline{45}\};$$
$$Q_3 = \{\overline{2}, \overline{6}, \overline{10}, \overline{14}, \overline{18}, \overline{22}, \overline{26}, \overline{30}, \overline{34}, \overline{38}, \overline{42}, \overline{46}\};$$
$$Q_4 = \{\overline{3}, \overline{7}, \overline{11}, \overline{15}, \overline{19}, \overline{23}, \overline{27}, \overline{31}, \overline{35}, \overline{39}, \overline{43}, \overline{47}\}.$$

Let $R = \{\overline{0}, \overline{8}, \overline{16}, \overline{24}, \overline{32}, \overline{40}\}$ be a subset of U, $\overline{Apr}(R) = Q_1 = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}, \overline{24}, \overline{28}, \overline{32}, \overline{36}, \overline{40}, \overline{44}\}$, and $\underline{Apr}(R) = \emptyset$. We will prove that R is a rough ring with binary operation $(+_{48}, \cdot_{48})$.

Table 1. Table *Cayley* $+_{48}$ on R

$+_{48}$	$\overline{0}$	$\overline{8}$	$\overline{16}$	$\overline{24}$	$\overline{32}$	$\overline{40}$
$\overline{0}$	$\overline{0}$	$\overline{8}$	$\overline{16}$	$\overline{24}$	$\overline{32}$	$\overline{40}$
$\overline{8}$	$\overline{8}$	$\overline{16}$	$\overline{24}$	$\overline{32}$	$\overline{40}$	$\overline{0}$
$\overline{16}$	$\overline{16}$	$\overline{24}$	$\overline{32}$	$\overline{40}$	$\overline{0}$	$\overline{8}$
$\overline{24}$	$\overline{24}$	$\overline{32}$	$\overline{40}$	$\overline{0}$	$\overline{8}$	$\overline{16}$
$\overline{32}$	$\overline{32}$	$\overline{40}$	$\overline{0}$	$\overline{8}$	$\overline{16}$	$\overline{24}$
$\overline{40}$	$\overline{40}$	$\overline{0}$	$\overline{8}$	$\overline{16}$	$\overline{24}$	$\overline{32}$

According to Table 1, the key points are as follows.

1. *For every $p_1, p_2 \in R$, $p_1 +_{48} p_2 \in \overline{Apr}(R)$.*
2. *For every $p_1, p_2, p_3 \in R$, $(p_1 +_{48} p_2) +_{48} p_3 = p_1 +_{48} (p_2 +_{48} p_3)$ in $\overline{Apr}(R)$.*
3. *There is $\overline{0} \in \overline{Apr}(R)$, then $p_1 +_{48} \overline{0} = \overline{0} +_{48} p_1 = p_1$ for every $p_1 \in R$.*
4. *According to Table 2, for every $p_1 \in R$, there is $p_1^{-1} \in R$, then $p_1 +_{48} p_1^{-1} = p_1^{-1} +_{48} p_1 = \overline{0}$.*

Table 2. Invers Table on R

a	$\overline{0}$	$\overline{8}$	$\overline{16}$	$\overline{24}$	$\overline{32}$	$\overline{40}$
a^{-1}	$\overline{0}$	$\overline{40}$	$\overline{32}$	$\overline{24}$	$\overline{16}$	$\overline{8}$

5. *For every $p_1, p_2 \in R$, then $p_1 +_{48} p_2 = p_2 +_{48} p_1$.*
6. *According to Table 3, for every $p_1, p_2 \in R$, then $p_1 \cdot_{48} p_2 \in \overline{Apr}(R)$.*
7. *For every $p_1, p_2, p_3 \in R$, then $(p_1 \cdot_{48} p_2) \cdot_{48} p_3 = p_1 \cdot_{48} (p_2 \cdot_{48} p_3)$ in $\overline{Apr}(R)$.*

Table 3. Table *Cayley* $\cdot_{48}$ on R

$\cdot_{48}$	$\overline{0}$	$\overline{8}$	$\overline{16}$	$\overline{24}$	$\overline{32}$	$\overline{40}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{8}$	$\overline{0}$	$\overline{16}$	$\overline{32}$	$\overline{0}$	$\overline{16}$	$\overline{32}$
$\overline{16}$	$\overline{0}$	$\overline{32}$	$\overline{16}$	$\overline{0}$	$\overline{32}$	$\overline{16}$
$\overline{24}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{32}$	$\overline{0}$	$\overline{16}$	$\overline{32}$	$\overline{0}$	$\overline{16}$	$\overline{32}$
$\overline{40}$	$\overline{0}$	$\overline{32}$	$\overline{16}$	$\overline{0}$	$\overline{32}$	$\overline{16}$

8. *For every $p_1, p_2, p_3 \in R$, then $(p_1 +_{48} p_2) \cdot_{48} p_3 = (p_1 \cdot_{48} p_3) +_{48} (p_2 \cdot_{48} p_2)$ in $\overline{Apr}(R)$ and $p_1 \cdot_{48} (p_2 +_{48} p_3) = (p_1 \cdot_{48} p_2) +_{48} (p_1 \cdot_{48} p_3)$ in $\overline{Apr}(R)$.*

It is proved that R is the rough ring of the approximation space $(\mathcal{S}, \varphi)$.

Example 2. *Using the approximation space in Example 1, let $W \subseteq R$ be a nonempty set with $W = \{\overline{0}, \overline{24}\}$, $\overline{Apr}(W) = Q_1 = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}, \overline{24}, \overline{28}, \overline{32}, \overline{36}, \overline{40}, \overline{44}\}$, and $\underline{Apr}(W) = \emptyset$.*

1. *$p_1 - p_2 \in \overline{Apr}(W)$, for every $p_1, p_2 \in W$;*
2. *$rp_1 \in \overline{Apr}(W)$ and $p_1 r \in \overline{Apr}(W)$, for every $p_1 \in W$ and $r \in R$.*

It is proven that W is a rough ideal of the rough ring R.

To construct new approximation spaces based on group structures and subgroup relations, we consider the following result.

Proposition 3.1. *Let $(\mathcal{S}, \varphi)$ be an approximation space, where $\mathcal{S}$ is a finite group, and define the equivalence relation φ by $p_1 \varphi p_2$ if and only if $p_1 - p_2 \in M$, for $p_1, p_2 \in \mathcal{S}$, where M is a subgroup of $\mathcal{S}$. Then there exists an approximation space $(\mathcal{S}', \varphi')$, where $\mathcal{S}' = \{p_1 + Q \mid p_1 \in \mathcal{S}\}$, for a subgroup Q of $\mathcal{S}$.*

Proof. We will show that there exists an approximation space $(\mathcal{S}', \varphi')$, where $\mathcal{S}' = \{p_1 + Q \mid p_1 \in \mathcal{S}\}$, for some subgroup Q of $\mathcal{S}$. Given an approximation space $(\mathcal{S}, \varphi)$, where $p_1 \varphi p_2$ if and only if $p_1 - p_2 \in M$, for $p_1, p_2 \in \mathcal{S}$, and M is a subgroup $\mathcal{S}$. The equivalence classes induced by φ are of the form $0 + M, p_1 + M, p_2 + M, ..., p_n + M$. Now choose $Q = 0 + M$ subgroup of $\mathcal{S}$, then $\mathcal{S}' = \{p_1 + Q \mid p_1 \in \mathcal{S}\}$. Since $0 + Q = Q \in \mathcal{S}'$, we have $\mathcal{S}' \neq \emptyset$. Next, we define a relation γ' on $\mathcal{S}'$ as follows: for any $p_1 + Q$, $p_2 + Q \in \mathcal{S}'$, we say $(p_1 + Q)\varphi'(p_2 + Q)$ if and only if $p_1 - p_2 \in \mathcal{S}$. We will show that the relation φ' is an equivalence relation on $\mathcal{S}'$.

1. For any $p_1 + Q \in \mathcal{S}'$, we have $p_1 - p_1 = 0 \in \mathcal{S}$, so $(p_1 + Q)\varphi'(p_1 + Q)$, i.e. φ' is reflexive.
2. For any $p_1 + Q$, $p_2 + Q \in \mathcal{S}'$, if $(p_1 + Q)\varphi'(p_2 + Q)$, then $p_1 - p_2 \in \mathcal{S}$, which implies $p_2 - p_1 = -(p_1 - p_2) \in \mathcal{S}$. So, $(p_2 + Q)\varphi'(p_1 + Q)$, i.e. φ' is symmetric.
3. For any $p_1 + Q$, $p_2 + Q$, $p_3 + Q \in \mathcal{S}'$ if $(p_1 + Q)\gamma'(p_2 + Q)$ and $(p_2 + Q)\varphi'(p_3 + Q)$, then $p_1 - p_2 \in \mathcal{S}$ and $p_2 - p_3 \in \mathcal{S}$. So, $(p_1 - p_2) + (p_2 - p_3) = p_1 - p_3 \in \mathcal{S}$, which implies $(p_1 + Q)\varphi'(p_3 + Q)$, i.e. φ' is transitive.

Thus, φ' is an equivalence relation on $\mathcal{S}'$. Therefore, $(\mathcal{S}', \varphi')$ is an approximation space. □

To construct new approximation spaces based on ring structures and ideal relations, we consider the following result.

Proposition 3.2. *Let $(\mathcal{S}, \varphi)$ be an approximation space, where $\mathcal{S}$ is a finite ring, and define the equivalence relation by $p_1 \varphi p_2$ if and only if $p_1 - p_2 \in P$, for $p_1, p_2 \in \mathcal{S}$, where P is an ideal of $\mathcal{S}$. Then there exists an approximation space $(\mathcal{S}', \varphi')$, where $\mathcal{S}' = \{p_1 + P \mid p_1 \in \mathcal{S}\}$, for an ideal P of $\mathcal{S}$.*

The steps of proving Proposition 3.2 are the same as the steps in the proof of Proposition 3.1.

Example 3. *Based on Example 1, using Proposition 3.2 we obtain the approximation space $(\mathcal{S}', \varphi')$, where $\mathcal{S}' = \{p_1 +_{48} Q_1 \mid p_1 \in \mathcal{S}\} = \{\overline{0} +_{48} Q_1, \overline{1} +_{48} Q_1, \overline{2} +_{48} Q_1, \overline{3} +_{48} Q_1\}$ with*

$$\overline{0} +_{48} Q_1 = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}, \overline{24}, \overline{28}, \overline{32}, \overline{36}, \overline{40}, \overline{44}\};$$
$$\overline{1} +_{48} Q_1 = \{\overline{1}, \overline{5}, \overline{9}, \overline{13}, \overline{17}, \overline{21}, \overline{25}, \overline{29}, \overline{33}, \overline{37}, \overline{41}, \overline{45}\}$$
$$\overline{2} +_{48} Q_1 = \{\overline{2}, \overline{6}, \overline{10}, \overline{14}, \overline{18}, \overline{22}, \overline{26}, \overline{30}, \overline{34}, \overline{38}, \overline{42}, \overline{46}\};$$
$$\overline{3} +_{48} Q_1 = \{\overline{3}, \overline{7}, \overline{11}, \overline{15}, \overline{19}, \overline{23}, \overline{27}, \overline{31}, \overline{35}, \overline{39}, \overline{43}, \overline{47}\}.$$

Define an equivalence relation φ' on the set $\mathcal{S}'$, for every $p_1 +_{48} Q_1$, $p_2 +_{48} Q_1 \in \mathcal{S}'$ holds $(p_1 +_{48} Q_1)\varphi'(p_2 +_{48} Q_1)$ if and only if $p_1 -_{48} p_2 \in \mathcal{S}$. The equivalence classes on the set U' are obtained as follows:

$$Q_1' = \{\overline{0} +_{48} Q_1, \overline{1} +_{48} Q_1, \overline{2} +_{48} Q_1, \overline{3} +_{48} Q_1\}.$$

Given a nonempty subset $R' \subseteq \mathcal{S}'$ where $R' = \{\overline{1} +_{48} Q_1, \overline{3} +_{48} Q_1\}$, $\overline{Apr}(R') = Q_1' = \{\overline{0} +_{48} Q_1, \overline{1} +_{48} Q_1, \overline{2} +_{48} Q_1, \overline{3} +_{48} Q_1\}$, and $\underline{Apr}(R') = \emptyset$.

The set R' is a rough ring with respect to the operations $+_{48}$ and $\cdot_{48}$ defined by:

$$(p_1 +_{48} Q_1) +_{48} (p_2 +_{48} Q_1) = (p_1 +_{48} p_2) +_{48} Q_1,$$
$$(p_1 +_{48} Q_1) \cdot_{48} (p_2 +_{48} Q_1) = (p_1 \cdot_{48} p_2) +_{48} Q_1,$$

for every $p_1 +_{48} Q_1$, $p_2 +_{48} Q_1 \in R'$. The set R' is called the rough quotient ring on the approximation space $(\mathcal{S}', \varphi')$.

Next, we construct the rough quotient module.

Example 4. *Let $(\mathcal{S}, \varphi)$ be an approximation space with $\mathcal{S} = \mathbb{Z}_{60}$. We define an equivalence relation φ such that $p_1 \varphi p_2$ holds if and only if $p_1 -_{60} p_2 = 3l$ with $p_1, p_2 \in \mathcal{S}$, and $l \in \mathbb{Z}$. The equivalence classes on the set $\mathcal{S}$ are obtained as follows:*
$Q_1 = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}, \overline{30}, \overline{33}, \overline{36}, \overline{39}, \overline{42}, \overline{45}, \overline{48}, \overline{51}, \overline{54}, \overline{57}\}$
$Q_2 = \{\overline{1}, \overline{4}, \overline{7}, \overline{10}, \overline{13}, \overline{16}, \overline{19}, \overline{22}, \overline{25}, \overline{28}, \overline{31}, \overline{34}, \overline{37}, \overline{40}, \overline{43}, \overline{46}, \overline{49}, \overline{52}, \overline{55}, \overline{58}\}$
$Q_3 = \{\overline{2}, \overline{5}, \overline{8}, \overline{11}, \overline{14}, \overline{17}, \overline{20}, \overline{23}, \overline{26}, \overline{29}, \overline{32}, \overline{35}, \overline{38}, \overline{41}, \overline{44}, \overline{47}, \overline{50}, \overline{53}, \overline{56}, \overline{59}\}$

Given a nonempty subset $R \subseteq \mathcal{S}$ with $R = \{\overline{4}, \overline{14}, \overline{18}, \overline{28}, \overline{32}, \overline{42}, \overline{46}, \overline{56}\}$, $\overline{Apr}(R) = Q_1 \cup Q_2 \cup Q_3 =$, and $\underline{Apr}(R) = \emptyset$. We will show that R is a rough ring with binary operation $(+_{60}, \cdot_{60})$.

1. *For every $p_1, p_2 \in R$, $p_1 +_{60} p_2 \in \overline{Apr}(R)$.*
2. *For every $p_1, p_2, p_3 \in R$, $(p_1 +_{60} p_2) +_{60} p_3 = p_1 +_{60} (p_2 +_{60} p_3)$ in $\overline{Apr}(R)$.*
3. *There is $e \in \overline{Apr}(R)$, then $p_1 +_{60} e = e +_{60} p_1 = p_1$ for every $p_1 \in R$.*
4. *For every $p_1 \in R$, there is $p_1^{-1} \in R$, then $p_1 +_{60} p_1^{-1} = p_1^{-1} +_{60} p_1 = e$.*
5. *For every $p_1, p_2 \in R$, then $p_1 +_{60} p_2 = p_2 +_{60} p_1$.*
6. *For every $p_1, p_2 \in R$, then $p_1 \cdot_{60} p_2 \in \overline{Apr}(R)$.*
7. *For every $p_1, p_2, p_3 \in R$, then $(p_1 \cdot_{60} p_2) \cdot_{60} p_3 = p_1 \cdot_{60} (p_2 \cdot_{60} p_3)$ in $\overline{Apr}(R)$.*
8. *For every $p_1, p_2, p_3 \in R$, then $(p_1 +_{60} p_2) \cdot_{60} p_3 = (p_1 \cdot_{60} p_3) +_{60} (p_2 \cdot_{60} p_3)$ in $\overline{Apr}(R)$ and $p_1 \cdot_{60} (p_2 +_{60} p_3) = (p_1 \cdot_{60} p_2) +_{60} (p_1 \cdot_{60} p_3)$ in $\overline{Apr}(R)$.*

Thus, R is a rough ring of the approximation space $(\mathcal{S}, \varphi)$.

Example 5. *Using the approximation space in Example 4, let $M \subseteq \mathcal{S}$ be a nonempty set with $M = \{\overline{0}, \overline{6}, \overline{12}, \overline{18}, \overline{24}, \overline{30}, \overline{36}, \overline{42}, \overline{48}, \overline{54}\}$, $\overline{Apr}(M) = Q_1 = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}, \overline{30}, \overline{33}, \overline{36}, \overline{39}, \overline{42}, \overline{45}, \overline{48}, \overline{51}, \overline{54}, \overline{57}\}$ and $\underline{Apr}(M) = \emptyset$. It will be proved that M is a commutative rough group with binary operation $+_{60}$.*

1. *For every $p_1, p_2 \in M$, $p_1 +_{60} p_2 \in \overline{Apr}(M)$.*
2. *For every $p_1, p_2, p_3 \in M$, $(p_1 +_{60} p_2) +_{60} p_3 = p_1 +_{60} (p_2 +_{60} p_3)$ in $\overline{Apr}(M)$.*
3. *There is $e \in \overline{Apr}(M)$, then $p_1 +_{60} e = e +_{60} p_1 = p_1$ for every $p_1 \in M$.*
4. *For every $p_1 \in M$, there is $p_1^{-1} \in M$, then $p_1 +_{60} p_1^{-1} = p_1^{-1} +_{60} p_1 = e$.*
5. *For every $p_1, p_2 \in M$, then $p_1 +_{60} p_2 = p_2 +_{60} p_1$.*

Hence, $\langle M, +_{60} \rangle$ is a commutative rough group. Next, it will be proved that M is a rough module over the rough ring R.

1. *For any $r \in R$ and $p_1, p_2 \in M$, then $r \cdot_{60} (p_1 +_{60} p_2) = (r \cdot_{60} p_1) +_{60} (r \cdot_{60} p_2)$.*
2. *For any $r_1, r_2 \in R$ and $p_1 \in M$, then $(r_1 +_{60} r_2) \cdot_{60} p_1 = (r_1 \cdot_{60} p_1) +_{60} (r_2 \cdot_{60} p_1)$.*
3. *For any $r_1, r_2 \in R$ and $p_1 \in M$, then $(r_1 \cdot_{60} r_2) \cdot_{60} p_1 = r_1 \cdot_{60} (r_2 \cdot_{60} p_1)$.*
4. *There exists $\overline{1} \in \overline{Apr}(R)$, such that for every $p_1 \in M$ holds $\overline{1} \cdot_{60} p_1 = p_1$ where $\overline{1}$ is the unit element of R.*

It is proved that M is a rough module over the rough ring R.

Example 6. *Using the approximation space in Example 4, let $N \subseteq M$ be a nonempty set with $N = \{\overline{0}, \overline{12}, \overline{24}, \overline{36}, \overline{48}\}$, $\overline{Apr}(N) = Q_1 = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}, \overline{30}, \overline{33}, \overline{36}, \overline{39}, \overline{42}, \overline{45}, \overline{48}, \overline{51}, \overline{54}, \overline{57}\}$ and $\underline{Apr}(N) = \emptyset$.*

1. *For every $n_1, n_2 \in N$, then $n_1 +_{60} n_2 \in \overline{Apr}(N)$.*
2. *For every $n \in N$, then $n^{-1} \in N$.*
3. *$r \cdot_{60} n \in \overline{Apr}(N)$, for every $r \in R, n \in N$.*

It is proven that N is a rough submodule of M over the rough ring R.

Proposition 3.3. *Let $(\mathcal{S}, \varphi)$ be an approximation space, where $\mathcal{S}$ is a finite module, and define the equivalence relation by $p_1 \varphi p_2$ if and only if $p_1 - p_2 \in K$, for $p_1, p_2 \in \mathcal{S}$, where K is a submodule of $\mathcal{S}$. Then there exists an approximation space $(\mathcal{S}', \varphi')$, where $\mathcal{S}' = \{p_1 + N | a \in \mathcal{S}\}$, for a submodule N of $\mathcal{S}$.*

The steps of proving Proposition 3.3 are the same as the steps in the proof Proposition 3.1.

Example 7. *Based on Example 4, using Proposition 3.3 we obtain the approximation space $(\mathcal{S}', \varphi')$, with $\mathcal{S}' = \{p_1 +_{60} Q_1 \mid p_1 \in \mathcal{S}\} = \{\overline{0} +_{60} Q_1, \overline{1} +_{60} Q_1, \overline{2} +_{60} Q_1\}$ with*
$\overline{0} +_{60} Q_1 = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}, \overline{30}, \overline{33}, \overline{36}, \overline{39}, \overline{42}, \overline{45}, \overline{48}, \overline{51}, \overline{54}, \overline{57}\}$;
$\overline{1} +_{60} Q_1 = \{\overline{1}, \overline{4}, \overline{7}, \overline{10}, \overline{13}, \overline{16}, \overline{19}, \overline{22}, \overline{25}, \overline{28}, \overline{31}, \overline{34}, \overline{37}, \overline{40}, \overline{43}, \overline{46}, \overline{49}, \overline{52}, \overline{55}, \overline{58}\}$;
$\overline{2} +_{60} Q_1 = \{\overline{2}, \overline{5}, \overline{8}, \overline{11}, \overline{14}, \overline{17}, \overline{20}, \overline{23}, \overline{26}, \overline{29}, \overline{32}, \overline{35}, \overline{38}, \overline{41}, \overline{44}, \overline{47}, \overline{50}, \overline{53}, \overline{56}, \overline{59}\}$.

Define an equivalence relation φ' on the set $\mathcal{S}'$, for every $p_1 +_{60} Q_1, p_2 +_{60} Q_1 \in \mathcal{S}'$ holds $(p_1 +_{60} Q_1)\varphi'(p_2 +_{60} Q_1)$ if and only if $p_1 -_{60} p_2 \in \mathcal{S}$. The equivalence classes on the set $\mathcal{S}'$ are obtained as follows:
$Q_1' = \{\overline{0} +_{60} Q_1, \overline{1} +_{60} Q_1, \overline{2} +_{60} Q_1\}$.

Let $M' \subseteq \mathcal{S}'$ be a nonempty subset with $M' = \{\overline{1} +_{60} Q_1, \overline{2} +_{60} Q_1\}$, $\overline{Apr}(M') = Q_1' = \{\overline{0} +_{60} Q_1, \overline{1} +_{60} Q_1, \overline{2} +_{60} Q_1\}$, $\underline{Apr}(M') = \emptyset$.

The set M' is a rough module over the rough ring R concerning the scalar multiplication operation $r \cdot_{60} (p_1 +_{60} Q_1) = (r \cdot_{60} p_1) +_{60} Q_1$, for any $r \in R$ and $p_1 +_{60} Q_1 \in M'$. The set M' is called the rough quotient module on the approximation space $(\mathcal{S}', \varphi')$.

4. CONCLUSIONS

Based on the results and discussions, it has been shown that the rough quotient group $\langle R/V, + \rangle$ forms a commutative rough group, where R is a rough ring and V is a rough ideal of R. The set $R/V = \{\overline{r} = r + \overline{Apr}(V) \mid r \in R\}$ is equipped with well-defined addition and multiplication operations, making it a rough ring referred to as rough quotient ring. Since V is a rough ideal, its upper approximation $\overline{Apr}(V)$ forms an ideal, implying that V is an arbitrary nonempty subset of some ideal E in a universe $\mathcal{S}$ such that $\overline{Apr}(V) = E$.

Furthermore, the rough quotient group $\langle M/K, + \rangle$ is also a commutative rough group, where M is a rough module over the rough ring R and K is a rough submodule of M. The scalar multiplication defined on M/K is well defined due to the structure of the submodule preserved by $\overline{Apr}(K)$. Therefore, M/N forms a rough quotient module over the rough ring R, where K is an arbitrary nonempty subset of a submodule E in a universe $\mathcal{S}$ such that $\overline{Apr}(N) = E$. These findings confirm the structural consistency of rough quotient rings and modules within the framework of rough set theory applied to algebraic systems over finite universes.

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