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Research Paper

The Cleanness Property of The Integers Modulo n ($\mathbb{Z}_n$)

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Abstract
Assume that R is a ring with identity. A ring R is said to be clean when each of its elements can be written as the sum of an idempotent and a unit element within the ring. A stronger condition, known as strongly clean, requires that these elements commute under multiplication. As a special case, a module M over ring R is called a clean module when the endomorphism ring of the M is a clean module over R. Moreover, when the ring endomorphism of R-module M is a strongly clean, then the module M is referred to as a strongly clean. We know that the integers modulo n, denoted by $\mathbb{Z}_n$, is a ring by the set of congruence classes modulo n, with standard addition and multiplication operations. In this study, we explore the cleanness properties of the ring $\mathbb{Z}_n$ and establish that it is a strongly clean ring. Furthermore, we study about the cleanness of $\mathbb{Z}_n$ as a module over $\mathbb{Z}$ and investigate the strongly cleanness of it module.

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1. INTRODUCTION

The property known as cleanness is well understood in the ring and module theory. The notion of a cean ring was initially proposed by W.K. Nicholson in 1997, a ring is called clean if each of its elements can be represented as the sum of an idempotent and a unit element [1]. In [2] has studied right clean rings, extending the idea of clean rings through the use of right unit elements, and in [3] have studied that field is a clean ring. Kosan et al. have explained the concept of weakly clean rings [4]. In addition, the exploration of the cleanness property of semirings has been studied in [5, 6]. Yuwaningsih et al. have studied the cleanness property of the ideal and the regular element of it [7]. The study of clean rings and their properties has been extensively studied, such as in [8, 9, 10].

On the other hand, when these unit and idempotent elements of clean elements commute under multiplication, the ring is known as a strongly clean ring [11]. In [12] have explored the new subclass of strongly clean rings and [13] had examined the conditions that affect the strongly clean property in rings. Cui and Wang have investigated the properties of strongly*-clean ring [14]. Das and Kar have studied the strongly clean property of semirings [15]. Moreover, many researchers have learned about strongly clean rings, as exemplified by [16].

The cleanness concept also extends to modules. An R-module M, it is possible to construct a homomorphism from M to itself, known as an endomorphism. The set of all endomorphisms, represented by $End_R(M)$, forms a ring with addition and composition functions operation [17]. Furthermore, the R-module M is regarded as clean if its endomorphism ring is a clean [18]. Similarly, when the endomorphism ring is strongly clean, the module M is called a strongly clean module [18, 19]. Krisnamurti has studied in the right clean modules [20]. Further investigations into clean modules are presented in studies such as [21, 22]. Meanwhile, the concept has also been extended to clean comodules and clean coalgebra, as discussed in [23, 24, 25].

Let $\mathbb{Z}_n$ be the set of integers modulo n, where n is a natural number. It is well known that $\mathbb{Z}_n$, equipped with addition and multiplication operations modulo n, forms a commutative ring with identity. When n is a prime, $\mathbb{Z}_n$ forms a field and its already known as a clean ring [3]. However, when n is a composite number, $\mathbb{Z}_n$ is not a field, and there is no general guarantee that it is clean. On the other hand, $\mathbb{Z}_n$ can also be viewed as a module over $\mathbb{Z}$. Furthermore, since $\mathbb{Z}_n$ is a commutative ring, then also satisfies the stronger property of being strongly clean. This study aims to prove that $\mathbb{Z}_n$ is a clean ring for every $n \in N$, as well as to investigate its cleanness as a $\mathbb{Z}$-module, and investigate the strongly cleanness of it module.

2. METHODS

This study employed a theoretical approach based on a literature review. The focus lies on investigating the cleanness property of $\mathbb{Z}_n$ both as a ring and as a module over $\mathbb{Z}$. The research procedure in this study involves exploring the concepts from ring and module theory, with a particular focus on unit elements, idempotent elements, endomorphism module, and the cleanness properties of rings and modules. This exploration was conducted through an extensive literature review by studying relevant theories from textbooks, i.e [26, 27, 28] and recent research articles about the cleanness property in rings and modules. The Figure 1 illustrates the summary of prior research and the current state of this research, which is presented below.

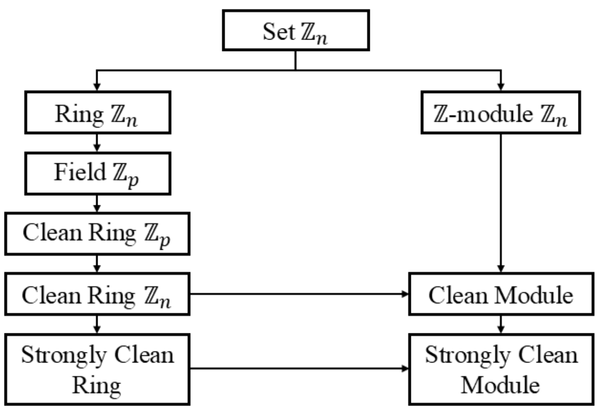

Figure 1. The Current State of Knowledge in this Research

3. RESULTS AND DISCUSSION

Given a ring $(\mathbb{Z}_n, +_n, \cdot_n)$ with $n \in \mathbb{N}$. Let $\mathbb{Z}_n$ represent the set of integers modulo a prime $p \in \mathbb{N}$. The ring is known to be commutative ring in which every nonzero element of $\mathbb{Z}_p$ have a multiplicative inverse. Consequently, $\mathbb{Z}_p$ forms a field. Conversely, when considering the ring $\mathbb{Z}_k$ be the ring where $k \in \mathbb{N}$ is composite, the structure fails to form a field due to the existence of nonzero elements lacking multiplicative inverses. Furthermore, for any integer n, the structure $\mathbb{Z}_n$ admits the intepretation of a $\mathbb{Z}$-module. In this study, we will discuss the cleanness property of $\mathbb{Z}_n$ both as a ring and a $\mathbb{Z}$-module. Additionally, the cleanness property of the matrix over $\mathbb{Z}_n$ will be investigated. We begin by exploring the cleanness property of the ring $\mathbb{Z}_n$.

3.1 The Cleanness Property in the Ring of $\mathbb{Z}_n$

Previously, it was explained that there is a difference between the ring $\mathbb{Z}_p$, with $p \in \mathbb{N}$ is assumed to be prime, and the ring $\mathbb{Z}_k$, where k is a composite number. The ring $\mathbb{Z}_p$ known as a field, and according to [3], a field is also a clean ring. The following lemma confirms the clean property of the ring $\mathbb{Z}_p$.

Lemma 1. *[3] For every prime p, the ring $\mathbb{Z}_p$ forms a field. Thus, $\mathbb{Z}_p$ qualifies as a clean ring.*

Proof. It will be proven that $\mathbb{Z}_p$ as a field, qualifies as a clean ring by expressing that each of its elements as the sum of an idempotent and a unit. Since the ring $\mathbb{Z}_p$ is a field, every nonzero elements serve as units, i.e. $U(\mathbb{Z}_p) = \{\bar{a} \in \mathbb{Z}_p \mid \bar{a} \neq \bar{0}\}$ and an idempotent element of $\mathbb{Z}_p$ are $\mathrm{Id}(\mathbb{Z}_p) = \{\bar{0}, \bar{1}\}$, then

(a) For $\bar{x} = \bar{0}$, the zero element:

$$\bar{x} = \bar{1} +_p \overline{p-1} = \overline{1+p-1} = \overline{1-1+p} = \bar{p} = \bar{0}.$$

Since every nonzero element in $\mathbb{Z}_p$ is a unit, there are $(p-\bar{1}) \in \mathbb{Z}_p$ is an unit elements. Moreover, $\bar{1}$ is an idempotent element. Therefore, the zero element $(\bar{0})$ is a clean element.

(b) For $\bar{x} \neq \bar{0}$, a nonzero element of $\mathbb{Z}_p$:

$$\bar{x} = \bar{0} +_p \bar{x}.$$

Since $\bar{x} \neq \bar{0}$, then $\bar{x} \in U(\mathbb{Z}_p)$ and we know that $\bar{0} \in \mathrm{Id}(\mathbb{Z}_p)$, it follows that every nonzero element $\bar{x} \in \mathbb{Z}_p$ satisfies the condition of being a clean element.

Based on (a) and (b), we conclude that every element $\bar{x} \in \mathbb{Z}_p$ is a clean element. As a result, the ring $\mathbb{Z}_p$ is a clean. □

As has been established, the ring $\mathbb{Z}_p$ is a field. Therefore, it is a commutative ring, and by the consequence of Lemma 1, it also satisfies the condition of strongly clean elements, as stated in the following lemma.

Lemma 2. *Given any prime p, the ring $\mathbb{Z}_p$ qualifies as a strongly clean.*

Proof. In Lemma 1, it has been proven that the ring $\mathbb{Z}_p$ is a clean. Moreover, as $\mathbb{Z}_p$ is a field, it has the commutative property of its multiplication operation "$\cdot_p$". Based on this, the condition of a strongly clean ring is satisfied, i.e., the multiplication operation between an idempotent and a unit is commutative. Hance, it follows taht the ring $\mathbb{Z}_p$ qualifies as strongly clean. □

It is well known that composite numbers can be expressed as the product of prime numbers. We will investigate the cleanness of the ring $\mathbb{Z}_k$, where k is a composite number.

Theorem 1. *Let $(\mathbb{Z}_k, +_k, \cdot_k)$ be a ring with unity element $\overline{1_k}$. For every composite number $k \in \mathbb{N}$, the ring $\mathbb{Z}_k$ is a clean.*

Proof. It is known that k is a composite number that can be expressed as the product of prime numbers. Based on this, we will prove that the ring $\mathbb{Z}_k$ is a clean ring by showing that

$$\mathbb{Z}_k \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}.$$

A mapping δ is defined as follows:

$$\delta : \mathbb{Z}_k \to \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}$$

$$\overline{a} \mapsto \delta(\overline{a}) = (a \bmod p_1, a \bmod p_2, \ldots, a \bmod p_i).$$

It should be recalled that, based on [29], it has been stated that the direct product formed from rings itself constitutes a ring. Thus, $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}$ forms a ring. It is shown that

$$\mathbb{Z}_k \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}$$

by proving that the mapping δ is a ring isomorphism.

(a) The mapping δ preserves the addition operation.
To prove that the mapping δ preserves the addition operation, we need to show that for every $\overline{b}, \overline{d} \in \mathbb{Z}_k$, we have

$$\delta(\overline{b} +_k \overline{d}) = \delta(\overline{b}) + \delta(\overline{d}).$$

LHS:

$$\begin{aligned}\delta(\overline{b} +_k \overline{d}) &= \delta(\overline{b+d}) \\ &= ((b+d) \bmod p_1, (b+d) \bmod p_2, \dots, \\ &\quad (b+d) \bmod p_i) \\ &= ([(b \bmod p_1) + (d \bmod p_1)] \bmod p_1, \\ &\quad [(b \bmod p_2) + (d \bmod p_2)] \bmod p_2, \\ &\quad \dots, \\ &\quad [(b \bmod p_i) + (d \bmod p_i)] \bmod p_i).\end{aligned}$$

RHS:

$$\begin{aligned}\delta(\overline{b}) + \delta(\overline{d}) &= (b \bmod p_1, b \bmod p_2, \dots, b \bmod p_i) + \\ &\quad (d \bmod p_1, d \bmod p_2, \dots, d \bmod p_i) \\ &= ([(b \bmod p_1) + (d \bmod p_1)] \bmod p_1, \\ &\quad [(b \bmod p_2) + (d \bmod p_2)] \bmod p_2, \\ &\quad \dots, \\ &\quad [(b \bmod p_i) + (d \bmod p_i)] \bmod p_i).\end{aligned}$$

By comparing both sides, we see that

$$\delta(\overline{b} +_k \overline{d}) = \delta(\overline{b}) + \delta(\overline{d}).$$

Therefore, it is concluded that the mapping δ preserves the addition operation from the ring $\mathbb{Z}_k$ to the ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}$.

(b) The mapping δ preserves the multiplication operation.
We investigate whether the mapping δ preserves multiplication by showing that for every $\bar{b}, \bar{d} \in \mathbb{Z}_k$, it holds that

$$\delta(\bar{b} \cdot_k \bar{d}) = \delta(\bar{b}) \cdot \delta(\bar{d}).$$

LHS:

$$\begin{aligned}\delta(\bar{b} \cdot_k \bar{d}) &= \delta(\overline{b \cdot d}) \\ &= \big((b \cdot d) \bmod p_1, (b \cdot d) \bmod p_2, \dots, (b \cdot d) \\ &\quad \bmod p_i\big) \\ &= \big([(b \bmod p_1) \cdot (d \bmod p_1)] \bmod p_1, \\ &\quad [(b \bmod p_2) \cdot (d \bmod p_2)] \bmod p_2, \dots, \\ &\quad [(b \bmod p_i) \cdot (d \bmod p_i)] \bmod p_i\big).\end{aligned}$$

RHS:

$$\begin{aligned}\delta(\bar{b}) \cdot \delta(\bar{d}) &= (b \bmod p_1, b \bmod p_2, \dots, b \bmod p_i) \cdot \\ &\quad (d \bmod p_1, d \bmod p_2, \dots, d \bmod p_i) \\ &= \big([(b \bmod p_1) \cdot (d \bmod p_1)] \bmod p_1, \\ &\quad [(b \bmod p_2) \cdot (d \bmod p_2)] \bmod p_2, \dots, \\ &\quad [(b \bmod p_i) \cdot (d \bmod p_i)] \bmod p_i\big).\end{aligned}$$

Based on the results of the LHS and RHS, we find that LHS = RHS. Therefore,

$$\delta(\bar{a} \cdot_k \bar{b}) = \delta(\bar{a}) \cdot \delta(\bar{b}).$$

This proves that the mapping δ preserves the multiplication operation from the ring $\mathbb{Z}_k$ to the ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}$.

(c) The mapping δ is an injective.
We will investigate that the mapping δ is injective by showing that $\ker(\delta) = \{\bar{0}\} \in \mathbb{Z}_k$. Given that $\bar{a} \in \ker(\delta)$, then

$$\delta(\bar{a}) = (a \bmod p_1, a \bmod p_2, \dots, a \bmod p_i) = (0, \dots, 0).$$

It follows that

$$a \bmod p_j = 0,$$

so that

$$a \equiv 0 \pmod{p_j}.$$

This means that $p_j \mid a$, for $j = 1, 2, \dots, i$. Furthermore, since the p_j are coprime factors of k, from the basic properties of integers, we have

$$k = p_1 p_2 \cdots p_i \mid a.$$

Thus,

$$a \equiv 0 \pmod{k},$$

which implies

$$\bar{a} = \bar{0} \in \mathbb{Z}_k.$$

Therefore, it was obtained that $\ker(\delta) = \{\bar{0}\}$. So, it is proven that the mapping δ is injective.

(d) The mapping δ is a surjective.
Prove that the mapping δ is surjective by showing that for every element

$$\big(\bar{b}_1, \bar{b}_2, \dots, \bar{b}_i\big) \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i},$$

there exists $\bar{a} \in \mathbb{Z}_k$ such that

$$\delta(\bar{a}) = \big(b_1 \bmod p_1, \dots, b_i \bmod p_i\big).$$

Given $\bar{a} \in \mathbb{Z}_k$, then

$$\delta(\bar{a}) = (a \bmod p_1, \dots, a \bmod p_i) = (b_1 \bmod p_1, \dots, b_i \bmod p_i).$$

Consequently, we get

$$a \bmod p_j = b_j \bmod p_j,$$

so that the following system of congruences is formed:

$$\begin{cases} a \equiv b_1 \pmod{p_1} \\ a \equiv b_2 \pmod{p_2} \\ \vdots \\ a \equiv b_i \pmod{p_i} \end{cases} \tag{1}$$

where the p_j are pairwise coprime. Based on the Chinese Remainder Theorem [30], the system of congruences (??) has a unique solution modulo. Thus, it is proven that the mapping δ is surjective.

Based on the results in steps (1) – (4), we see that the mapping δ is a homomorphism exhibiting injectivity and surjectivity. Hence, the homomorphism δ is an isomorphism, which implies that $\mathbb{Z}_k \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}$. According to Lemma 1, it was stated that for any prime $p \in \mathbb{N}$, the ring $\mathbb{Z}_p$ is a clean. On the other hand, as explained in [31], the Cartesian product of clean rings is also a clean ring. Therefore, it can be concluded that the ring $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}$ is a clean ring. Furthermore, since $\mathbb{Z}_k \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}$, consequently, the ring $\mathbb{Z}_k$ is also a clean. Thus, it has been proven that for any composite number $k \in \mathbb{N}$, $\mathbb{Z}_k$ can be classified as a clean ring. ☐

By Lemma 2, it has been shown that the ring $\mathbb{Z}_p$ is a strongly clean, and as a consequence, we will identify the strongly clean property on $\mathbb{Z}_k$.

Theorem 2. *For every composite number k, the ring $\mathbb{Z}_k$ is a strongly clean.*

Proof. Using the same analogy as in Theorem 1, we obtain that $\mathbb{Z}_k \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}$. Referring to Lemma 2, which establishes that each $\mathbb{Z}_p$ is a strongly clean, its can deduce that the direct product $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}$ also satisfies the strongly clean property. Consequently, it can be conclude that the ring $\mathbb{Z}_k$ is a strongly clean. ☐

Based on Lemma 1, Lemma 2, Theorem 1, and Theorem 2, we conclude that both $\mathbb{Z}_p$ for prime p and $\mathbb{Z}_k$ for composite k there are clean and strongly clean rings, which gives rise to the following corollary.

Corollary 1. *For every natural number $n \in N$, the ring $\mathbb{Z}_n$ is both a clean and a strongly clean.*

In the following theorem, we investigate the cleanness properties of the image of a homomorphism from the clean ring $\mathbb{Z}_n$.

Theorem 3. *Let $\mathbb{Z}_n$ be a strongly clean ring, and let R be any ring. If there exists a ring homomorphism $j : \mathbb{Z}_n \to R$, then the image of j, denote by $\mathrm{Im}(j)$, is also a strongly clean.*

Proof. According to Corollary 1, it is already known that $\mathbb{Z}_n$ is a clean ring. Then, for every $\bar{a} \in \mathbb{Z}_n$, there exist a unit element $\bar{u}$ and an idempotent element $\bar{e}$ such that $\bar{a} = \bar{e} +_n \bar{u}$. By the definition of a ring homomorphism j, it holds that

$$j(\bar{a}) = j(\bar{e} +_n \bar{u}) = j(\bar{e}) + j(\bar{u}).$$

Since j is a homomorphism, we have the following:

(a) For every $\bar{u} \in U(\mathbb{Z}_n)$, it holds that $j(\bar{u})$ is a unit element in $\mathrm{Im}(j)$. We know that $\bar{u} \in U(\mathbb{Z}_n)$, so there exists $w \in \mathbb{Z}_n$ such that

$$j(\bar{u}) \cdot j(\bar{w}) = j(\bar{u} \cdot_n \bar{w}) = j(\bar{1}) = 1_R.$$

(b) For every $\bar{e} \in \mathrm{Id}(\mathbb{Z}_n)$, it holds that $j(\bar{e})$ is an idempotent element in $\mathrm{Im}(j)$, because

$$j(\bar{e}) \cdot j(\bar{e}) = j(\bar{e} \cdot_n \bar{e}) = j(\bar{e}).$$

Since $\mathrm{Im}(j)$ is a subring of R, it naturally inherits the ring structure. Moreover, every element in $\mathrm{Im}(j)$ can be written as the sum of an idempotent and a unit. Therefore, $\mathrm{Im}(j)$ qualifies as a clean ring. Recall that the ring $\mathbb{Z}_n$ is a strongly clean, the equality $\overline{ue} = \overline{eu}$ holds. Then we have the following result in $\mathrm{Im}(j)$:

$$j(\bar{u}) \cdot j(\bar{e}) = j(\overline{ue}) = j(\overline{eu}) = j(\bar{e}) \cdot j(\bar{u}).$$

Thus, the clean ring $\mathrm{Im}(j)$ is a strongly clean. ☐

3.2 The Cleanness Property in the Ring Matrix over $\mathbb{Z}_n$

Since $\mathbb{Z}_n$ is known to be a clean ring, we examine the cleanness of the matrix ring $M_m(\mathbb{Z}_n)$.

Theorem 4. *Let $(\mathbb{Z}_n, +_n, \cdot_n)$ be a ring, for every natural number $n \in \mathbb{N}$. Then the ring $M_m(\mathbb{Z}_n)$ is a clean.*

Proof. Referring to [1], it is explained that if the ring R is a clean, then $M_m(R)$ is also a clean. Since in Corollary 1 it has been proven that the ring $\mathbb{Z}_n$ is a clean, this implies that $M_m(\mathbb{Z}_n)$ is a clean. ☐

Theorem 5. *Let $(\mathbb{Z}_n, +_n, \cdot_n)$ be a ring, for every natural number $n \in \mathbb{N}$. Then the ring $M_2(\mathbb{Z}_n)$ qualifies as a strongly clean.*

Proof. According to [32], it has been known that $M_2(\mathbb{Z}_p)$ exhibits strongly cleanness property for every prime p. Based on Theorem 1, we have the ring isomorphism: $\mathbb{Z}_k \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}$, for every composite number k, where the p_j are pairwise coprime prime factors of k. Consequently, we have:

$$\begin{aligned} M_2(\mathbb{Z}_k) &= M_2\left(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}\right) \\ &\cong M_2(\mathbb{Z}_{p_1}) \times M_2(\mathbb{Z}_{p_2}) \times \cdots \times M_2(\mathbb{Z}_{p_i}). \end{aligned}$$

Since each $M_2(\mathbb{Z}_p)$ is strongly clean, and the direct product of strongly clean rings is also strongly clean, it follows that $M_2(\mathbb{Z}_k)$ is strongly clean. Therefore, since both $M_2(\mathbb{Z}_k)$ and $M_2(\mathbb{Z}_p)$ are strongly clean, we conclude that $M_2(\mathbb{Z}_n)$ is also strongly clean. ☐

3.3 The Cleanness Property in the $\mathbb{Z}$-Module $\mathbb{Z}_n$

Since $\mathbb{Z}_n$ is known to be a clean ring, we will investigate that $\mathbb{Z}_n$, regarded as a module over $\mathbb{Z}$, qualifies as a clean module as well.

Theorem 6. *Let $\mathbb{Z}_n$ be an abelian group and $\mathbb{Z}$ be a ring with unity. If $\mathbb{Z}_n$ is regarded as a module over $\mathbb{Z}$, then the $\mathbb{Z}$-module $\mathbb{Z}_n$ is a clean module.*

Proof. We aim to prove that the $\mathbb{Z}$-module $\mathbb{Z}_n$ is a clean by showing that the ring $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_n)$ is a clean ring. Define a function ξ as follows:

$$\xi : \mathbb{Z}_n \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_n), \quad \overline{a} \mapsto \xi(\overline{a}) = j_a,$$

with

$$j_a : \mathbb{Z}_n \to \mathbb{Z}_n$$

$$\overline{x} \mapsto j_a(\overline{x}) = \overline{ax} = ax \bmod n.$$

(a) The function ξ preserves addition.
Suppose $\overline{b}, \overline{d} \in \mathbb{Z}_n$, then

$$\xi(\overline{b} +_n \overline{d}) = j_{b+d}.$$

Given $\overline{x} \in \mathbb{Z}_n$, we have

$$\begin{aligned} j_{b+d}(\overline{x}) &= \overline{(b+d)x} \\ &= \overline{bx + dx} \\ &= \overline{bx} +_n \overline{dx} \\ &= j_b(\overline{x}) +_n j_d(\overline{x}). \end{aligned}$$

Hence, for all $\overline{x} \in \mathbb{Z}_n$, we get $j_{b+d}(\overline{x}) = j_b(\overline{x}) +_n j_d(\overline{x})$, which implies $\xi(\overline{b} +_n \overline{d}) = \xi(\overline{b}) +_n \xi(\overline{d})$. Thus, ξ preserves addition.

(b) The function ξ preserves multiplication.
We show that for every $\overline{b}, \overline{d} \in \mathbb{Z}_n$, then

$$\xi(\overline{b} \cdot_n \overline{d}) = \xi(\overline{bd}) = j_{bd}.$$

Given any $\overline{x} \in \mathbb{Z}_n$, it follows that

$$\begin{aligned} j_{bd}(\overline{x}) &= \overline{(bd)x} \\ &= \overline{b(dx)} \\ &= j_b(\overline{dx}) \\ &= j_b(j_d(\overline{x})) \\ &= (j_b \circ j_d)(\overline{x}). \end{aligned}$$

Hence, $\xi(\overline{b} \cdot_n \overline{d}) = \xi(\overline{b}) \circ \xi(\overline{d})$. Thus, ξ preserves multiplication.

(c) The function ξ is injective.
We show that $\ker(\xi) = \{\overline{0}\}$. Suppose $\overline{b} \in \ker(\xi)$, then

$$\xi(\overline{b}) = f_b = 0.$$

Take any $\overline{x} \in \mathbb{Z}_n$, so

$$j_a(\overline{x}) = \overline{bx} = \overline{0} \Rightarrow bx \equiv 0 \pmod{n}.$$

Since this holds for all $\overline{x} \in \mathbb{Z}_n$, we take $\overline{x}$ such that $\gcd(x, n) = 1$, so x^{-1} exists. Then

$$bx \equiv 0 \pmod{n} \Rightarrow b \equiv 0 \pmod{n} \Rightarrow \overline{b} = \overline{0}.$$

Therefore, $\ker(\xi) = \{\overline{0}\}$, so ξ is injective.

(d) The function ξ is surjective.
We show that for any $j \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_n)$, there exists $\overline{a} \in \mathbb{Z}_n$ such that $\xi(\overline{b}) = j_b$. Let $j \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_n)$, since j is a $\mathbb{Z}$-module homomorphism, it holds that

$$j = j \circ I_{\mathbb{Z}_n}.$$

Given $\overline{x} \in \mathbb{Z}_n$, which implies that

$$j(\overline{x}) = j(I_{\mathbb{Z}_n}(\overline{x})) = j(\overline{1} \cdot_n \overline{x}) = j(\overline{1}) \cdot_n \overline{x}.$$

Thus, for any $\overline{x} \in \mathbb{Z}_n$,

$$j(\overline{x}) = j(\overline{1}) \cdot_n \overline{x} = j_{j(\overline{1})}(\overline{x}).$$

So $j = j_b$ with $\overline{b} = j(\overline{1})$, i.e.,

$$j = \xi(\overline{b}).$$

Hence, ξ is surjective.
Referring to parts (a) through (d), we conclude that the function ξ defines a ring isomorphism. Therefore, $\mathbb{Z}_n \cong \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_n)$. Since $\mathbb{Z}_n$ is a clean ring, then $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_n)$ is also clean. Hence, $\mathbb{Z}_n$ satisfies the conditions of a clean module over $\mathbb{Z}$. □

As a consequence of Corollary 1, we further investigate the strongly clean property of $\mathbb{Z}_n$ as a $\mathbb{Z}$-module.

Theorem 7. *Given that $\mathbb{Z}_n$ is an abelian group and $\mathbb{Z}$ is a ring with unity. If $\mathbb{Z}_n$ is regarded as a $\mathbb{Z}$-module, then the $\mathbb{Z}$-module $\mathbb{Z}_n$ is a strongly clean.*

Proof. Given the same analogy as in Theorem 6, it is found that $\mathbb{Z}_n \cong \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_n)$. Referring to Corollary 1, since the ring $\mathbb{Z}_n$ is a strongly clean, this impiles that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_n)$ inherits the strongly clean property. Consequently, the $\mathbb{Z}$-module $\mathbb{Z}_n$ is a strongly clean. □

4. CONCLUSIONS

In this study, we have shown that the ring $\mathbb{Z}_n$ is clean, not only in the case when a n is a prime, where $\mathbb{Z}_n$ forms a field, but also when n is a composite. It has already been demonstrated that $\mathbb{Z}_n$ for n a composite number is isomorphic to the direct product of the ring $\mathbb{Z}_{(p_i)}$, where each p_i is a prime divisor of n. Since a finite direct product of clean rings is also clean, this leads to conclusion that the ring $\mathbb{Z}_n$ remains clean even for composite n. Furthermore, due to the isomorphic relationship between the endomorphism ring of the $\mathbb{Z}$-module $\mathbb{Z}_n$ and the ring $\mathbb{Z}_n$, the module inherits the clean property. Additionally, because $\mathbb{Z}_n$ is a commutative ring, its unit and idempotent elements commute. This implies that $\mathbb{Z}_n$ is not only clean but strongly clean, and consequently, the module $\mathbb{Z}_n$ is also strongly clean. Moreover, the cleanness property also applies to $M_m(\mathbb{Z}_n)$, for every $m, n \in N$. However, it is certain that $M_2(\mathbb{Z}_n)$ is strongly clean.

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REFERENCES

[1] W Keith Nicholson. Lifting idempotents and exchange rings. *Transactions of the American Mathematical Society*, 229:269–278, 1977.

[2] Cyrenia Novella Krisnamurti. Ring bersih kanan. *AdMathEdu: Jurnal Ilmiah Pendidikan Matematika, Ilmu Matematika dan Matematika Terapan*, 5(1), 2015 (in Indonesia).

[3] WK Nicholson and Yiqiang Zhou. Clean general rings. *Journal of Algebra*, 291(1):297–311, 2005.

[4] Tamer Koşan, Serap Sahinkaya, and Yiqiang Zhou. On weakly clean rings. *Communications in Algebra*, 45(8):3494–3502, 2017.

[5] S Kar and D Das. Clean semiring. *Beiträge Zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 64(1):197–207, 2023.

[6] Sondos Osama Hussain Jamal. *Clean Like Semiring Notions and Trivial Semiring Extension*. PhD thesis, Doctoral Dissertation, 2020.

[7] Dian Ariesta Yuwaningsih, Indah Emilia Wijayanti, and Budi Surodjo. On r-clean ideals. *Palestine Journal of Mathematics*, 12(2), 2023.

[8] Hongbo Zhang and Victor Camillo. On clean rings. *Communications in Algebra*, 44(6):2475–2481, 2016.

[9] Yuanlin Li and Qinghai Zhong. On clean, weakly clean and feebly clean commutative group rings. *Journal of Algebra and Its Applications*, 21(05):2250085, 2022.

[10] Nitin Arora and S Kundu. Commutative feebly clean rings. *Journal of Algebra and its Applications*, 16(07):1750128, 2017.

[11] W Keith Nicholson. Strongly clean rings and fitting's lemma. *Communications in Algebra*, 27(8):3583–3592, 1999.

[12] Orhan Gurgun and Sait Halicioglu Ungor. A subclass of strongly clean rings. *Communications in Mathematics*, 23(1):13–31, 2015.

[13] Janez Šter. Examples of strongly clean rings. *Communications in Algebra*, 47(11):4684–4696, 2019.

[14] Jian Cui and Zhou Wang. A note on strongly*-clean rings. *Journal of the Korean Mathematical Society*, 52(4):839–851, 2015.

[15] D Das and S Kar. Strongly clean semiring. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, 94(2):249–258, 2024.

[16] Pace Nielsen and Janez Šter. Connections between unit-regularity, regularity, cleanness, and strong cleanness of elements and rings. *Transactions of the American Mathematical Society*, 370(3):1759–1782, 2018.

[17] Gangyong Lee, Cosmin S Roman, and Xiaoxiang Zhang. Modules whose endomorphism rings are division rings. *Communications in Algebra*, 42(12):5205–5223, 2014.

[18] Victor P Camillo, Dinesh Khurana, TY Lam, WK Nicholson, and Y Zhou. Continuous modules are clean. *Journal of Algebra*, 304(1):94–111, 2006.

[19] Hongbo Zhang. On strongly clean modules. *Communications in Algebra*, 37(4):1420–1427, 2009.

[20] Cyrenia Novella Krisnamurti. Modul bersih kanan right clean modules. *Jurnal Penelitian*, 21(1):22–25, 2017 (in Indonesia).

[21] A Ismarwati, H France-Jackson, Sri Wahyuni, and IE Wijayanti. On nice modules. *JP Journal of Algebra, Number Theory and Applications*, 38(3):213, 2016.

[22] Yasser Ibrahim and Mohamed Yousif. Utumi modules. *Communications in Algebra*, 46(2):870–886, 2018.

[23] Nikken Prima Puspita, IE Wijayanti, and Budi Surodjo. Clean coalgebras and clean comodules of finitely generated projective modules. *Algebra and Discrete Mathematics*, 31(2):251–260, 2021.

[24] Nikken Prima Puspita, Indah Emilia Wijayanti, and Budi Surodjo. Some sufficient conditions of clean comodules. *The Aligarh Bulletin of Mathematics*, 41(2):1–20, 2022.

[25] Nikken Prima Puspita, Indah Emilia Wijayanti, and Budi Surodjo. Graded modules as a clean comodule. *Journal of Mathematics Research*, 12:66–73, 2020.

[26] Sri Wahyuni, Indah Emilia Wijayanti, Dian Ariesta Yuwaningsih, and Ari Dwi Hartanto. *Teori Ring dan Modul*. UGM PRESS, 2021.

[27] Robert Wisbauer. *Foundations of Module and Ring Theory*. Routledge, 2018.

[28] William A Adkins and Steven H Weintraub. *Algebra: An Approach Via Module Theory*, volume 136. Springer Science & Business Media, 2012.

[29] DS Dummit and RM Foote. Abstract algebra (2004), hoboken.

[30] Adillah Azzahra, Astuti Astuti, Adityawarman Hidayat, and Zulhendri Zulhendri. Chines remainder theorem (teorema sisa china). *Integrasi Episdemik: Jurnal Multidisiplin Inovatif*, 1(1):10–15, 2025.

[31] Garima Sharma and Amit B Singh. Strongly r-clean rings. *International Journal of Mathematics and Computer Science*, 13(2):207–214, 2018.

[32] Jianlong Chen, Xiande Yang, and Yiqiang Zhou. On strongly clean matrix and triangular matrix rings. *Communications in Algebra*, 34(10):3659–3674, 2006.